Geometric stochastic heat equations

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Abstract

We consider a natural class of $\mathbf{R}^d$-valued one-dimensional stochastic PDEs driven by space-time white noise that is formally invariant under the action of the diffeomorphism group on $\mathbf{R}^d$. This class contains in particular the KPZ equation, the multiplicative stochastic heat equation, the additive stochastic heat equation, and rough Burgers-type equations. We exhibit a one-parameter family of solution theories with the following properties:

1. For all SPDEs in our class for which a solution was previously available, every solution in our family coincides with the previously constructed solution, whether that was obtained using Itô calculus (additive and multiplicative stochastic heat equation), rough path theory (rough Burgers-type equations), or the Hopf-Cole transform (KPZ equation).
2. Every solution theory is equivariant under the action of the diffeomorphism group, i.e. identities obtained by formal calculations treating the noise as a smooth function are valid.
3. Every solution theory satisfies an analogue of Itô's isometry.
4. The counterterms leading to our solution theories vanish at points where the equation agrees to leading order with the additive stochastic heat equation.

In particular, points 2 and 3 show that, surprisingly, our solution theories enjoy properties analogous to those holding for both the Stratonovich and Itô interpretations of SDEs *simultaneously*. For the natural noisy perturbation of the harmonic map flow with values in an arbitrary Riemannian manifold, we show that all these solution theories coincide. In particular, this allows us to conjecturally identify the process associated to the Markov extension of the Dirichlet form corresponding to the L^2-gradient flow for the Brownian loop measure.

Keywords: Brownian loops, renormalisation, stochastic PDE

MSC classification: 60H15

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1 Introduction

One of the main goals of the present article is to build "the" most natural stochastic process taking values in the space of loops in a compact Riemannian manifold $\mathscr{M}$, i.e. a random map $u\colon S^1 \times \mathbf{R}_+ \to \mathscr{M}$. For a candidate to qualify for such an admittedly subjective designation, one would like it to be as "simple" as possible, all the while having as many nice properties as possible. In this article, we interpret this as follows.

1. The process u should be specified by only using the Riemannian structure on $\mathscr{M}$ and no additional data.
2. The process u should have a purely local specification in the sense that it satisfies the space-time Markov property.

3. It should be the unique process with these properties among a 'large' class of processes.

A natural way of constructing a candidate would be as follows. Let μ be the measure on $L_{\mathscr{M}} = \mathscr{C}(S^1, \mathscr{M})$ given by the law of a Brownian loop, i.e. the Markov process with generator the Laplace-Beltrami operator on $\mathscr{M}$, conditioned to return to its starting point at some fixed time (say 1). We fix its law at time 0 to be the probability measure on $\mathscr{M}$ with density proportional to $p_1(x,x)$, which guarantees that μ is invariant under rotations of the circle S^1. One can then consider the Dirichlet form $\mathscr{E}$ given for suitably smooth functions $\Phi\colon L_{\mathscr{M}} \to \mathbf{R}$ by

$$\mathscr{E}(\Phi, \Phi) = \int_{L_{\mathscr{M}}} \int_{S^1} \langle \nabla_x \Phi(u), \nabla_x \Phi(u) \rangle_{u(x)}\, dx\, \mu(du)\;,$$

where $\nabla_x \Phi(u) \in T_{u(x)}\mathscr{M}$ denotes the functional gradient of Φ and $\langle \cdot, \cdot \rangle_u$ denotes the scalar product in $T_u\mathscr{M}$. Unfortunately, while it is possible to show that $\mathscr{E}$ is regular and closable [RWZZ17], so that we can construct a (possibly non-conservative) Markov process from it, the uniqueness of Markov extensions for $\mathscr{E}$ is a hard open problem. Accordingly, point 3 above fails to be verifiable with this approach and it appears very difficult to extract any qualitative properties of the corresponding process, which is somewhat unsatisfactory.

Instead, we will directly make sense of the corresponding stochastic partial differential equation. At the formal (highly non-rigorous!) level, $\mathscr{E}$ is expected to represent the Langevin equation for the measure μ which, again at a purely heuristic level (but see [IM85, AD99] for rigorous interpretations of this identity), is given by

$$\mu(du) \propto e^{-H(u)} du\;, \quad H(u) = \frac{1}{2} \int_{S^1} g_{u(x)}(\partial_x u, \partial_x u)\, dx\;, \tag{1.1}$$

where "du" denotes the non-existent "Lebesgue measure" on the space of all loops with values in $\mathscr{M}$.

Remark 1.1 There are various results and heuristic calculations suggesting that the "correct" formal expression for the Brownian loop measure should be given by (1.1), but with H corrected by a suitable multiple of the integral of the scalar curvature along the trajectory u. We will revisit this point in more detail in Remark 1.13 and Section 4.3. To some extent, our results explain the appearance of this term, as well as why there is no consensus in the literature as to the "correct" constant multiplying it.

At this point we note that the L^2-gradient flow for H (in the intrinsic L^2 metric determined by g) is given by the classical curve-shortening flow (a particular instance of Eells-Sampson's harmonic map flow [ES64]) which, in local coordinates, can be written as

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma\;,$$

where Γ denotes the Christoffel symbols for the Levi-Civita connection and we use Einstein's convention of summation over repeated indices. Given that we obtained it as an L^2-gradient flow, the natural way of adding noise to this equation is to add white noise with a covariance structure that reflects the Riemannian structure of $\mathscr{M}$. In other words, we would like to consider the $\mathscr{M}$-valued SPDE formally given by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\,\xi_i\;, \tag{1.2}$$

where the ξ_i are i.i.d. space-time white noises and the σ_i are a collection of vector fields on $\mathscr{M}$ that are related to the (inverse) metric tensor by the identity

$$\sigma_i^\alpha(u)\,\sigma_i^\beta(u) = g^{\alpha\beta}(u)\;. \tag{1.3}$$

(Actually, the invariant measure for (1.2) would, at least formally, be given by (1.1) without the factor $\frac{1}{2}$, which then corresponds to the Brownian loop measure running at half of its natural speed. We stick to (1.2) as written in order to avoid having this extra factor 2 appearing throughout the article.) In (1.3) and throughout the paper, we adopt the following convention: every time we have a product of two factors containing an index i, a summation over i is implied. For instance:

$$\sigma_i^\alpha\,\sigma_i^\beta := \sum_i \sigma_i^\alpha\,\sigma_i^\beta,\quad \sigma_i^\alpha\,dW_i := \sum_i \sigma_i^\alpha\,dW_i,\quad \sigma_i^\beta\,\partial_\beta\sigma_i^\alpha := \sum_i \sigma_i^\beta\,\partial_\beta\sigma_i^\alpha.$$

In this article, we will mainly study systems of equations of the type (1.2) without assuming any relation between the functions $\Gamma^\alpha_{\beta\gamma}$ and σ_i^α. However, we will see that in the particular geometric case mentioned above additional cancellations take place (see Lemma 4.9 below) which then allow us to assign in a canonical way one specific Markov process on loop space to every equation of the type (1.2) in a way that preserves the natural symmetries of this class of equations and such that the counterterm added to a smoothened version of (1.2) is 'minimal' in a precise sense (see the list of properties on Page 6). It turns out that with this choice of solution theory, we expect "the" natural process we are after (the one that furthermore admits the Brownian loop measure as its invariant measure) to be given not by (1.2), but by the equation with an additional term $\frac{1}{8}\nabla R(u)$ added to the right hand side, where R denotes the scalar curvature of $\mathscr{M}$. More details are given in Conjecture 4.5 below, which in fact covers a much larger class of equations. The only reason why part of our results is not stated as a theorem is that we are missing an analogue of the result [CH16] in the context of discrete regularity structures [EH17], although such a result is widely expected to hold (see for example [HM18a, CGP17] for special cases).

Note that [Fun92] first studied a version of equation (1.2) where the i.i.d. noises ξ_i are white in time but coloured in space, and the stochastic integral is in the Stratonovich sense. The version of (1.2) with space-time white noise had to wait the invention of regularity structures before it could be properly understood and solved. We also recall that a natural process on loop space leaving invariant the Brownian loop measure on a Riemannian manifold was constructed by a number of authors in the nineties, see for example [Dri92, DR92, Dri94, ES96, Nor98]. The process constructed there is different from ours since, in the particular case of $\mathscr{M} = \mathbf{R}^d$, it would correspond to the Ornstein-Uhlenbeck process from Malliavin calculus, rather than the stochastic heat equation. In particular, it is not "local" in the sense that its driving noise has non-trivial spatial correlations. In terms of its interpretation as the Langevin equation associated to (1.1), our construction corresponds to taking gradients with respect to the intrinsic L^2 metric determined by the manifold, while previous results corresponded to taking gradients in the metric given by the H^1 norm.

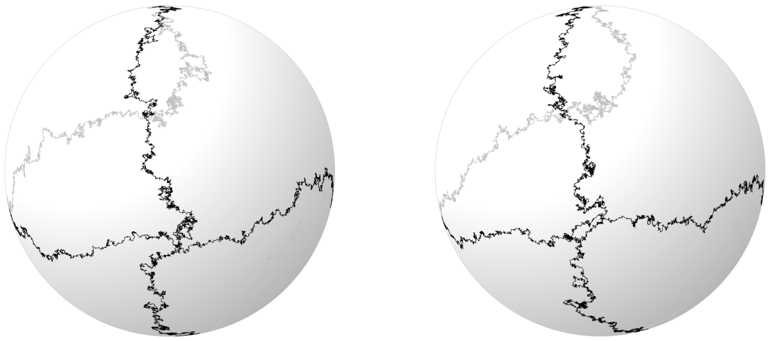

Figure 1: Solution to (1.2) on the sphere at two successive times. In this case, all processes in the canonical family agree since the scalar curvature is constant. We see that the global structure hasn't changed much, but the local structure is very different between the two times.

1.1 Informal overview of main results

The recently developed theory of regularity structures [Hai14] provides a tool to give meaning to equations of the type (1.2) (see the series of works [BHZ18, CH16, BCCH17] for a 'black box theorem' that applies in this case), but instead of giving a single notion of solution it gives a canonical *family* of notions of solution parametrised by a suitable "renormalisation group". In the problem under consideration, this group is quite large: it can be identified with $\mathcal{S} \approx (\mathbf{R}^{54}, +)$, even after taking into account simplifications arising from Gaussianity, the fact that the noises are i.i.d., and the $x \leftrightarrow -x$ symmetry. The general theory then in principle yields a 54-dimensional family of candidate solution theories parametrised by $\mathcal{S}$. Furthermore, this parametrisation is not canonical in general.

This should be contrasted with the case of SDEs with smooth coefficients where one has a one-parameter family of solution theories (interpolating between solutions in the Itô sense and solutions in the Stratonovich sense) and there are two distinguished points on that line. This can be formulated as follows. Write $U_c(\sigma, h)$ for the map sending initial conditions to the law of the (Itô) solution to

$$dx^\alpha = h^\alpha(x)\,dt + \sigma_i^\alpha(x)\,dW_i(t) + c\sigma_i^\beta(x)\,\partial_\beta\sigma_i^\alpha(x)\,dt\;, \tag{1.4}$$

with implicit summation over repeated indices and x taking values in $\mathbf{R}^d$. The following is then well-known.

1. The solution theory $U^{\text{Itô}} \stackrel{\text{def}}{=} U_0$ is the only solution theory (among this one-parameter family) such that $U_c(\sigma, h) = U_c(\bar\sigma, h)$, whenever $\sigma_i^\alpha \sigma_i^\beta = \bar\sigma_i^\alpha \bar\sigma_i^\beta$. We will refer to this property by saying that "$U^{\text{Itô}}$ satisfies Itô's isometry".
2. The solution theory $U^{\text{geo}} \stackrel{\text{def}}{=} U_{1/2}$ is the only one such that its arguments transform like vector fields under changes of coordinates. We will refer to this property by saying that "U^{geo} is equivariant under changes of coordinates".

Of course the parametrisation of the family $c \mapsto U_c$ of solution theories is completely arbitrary: we could have interpreted (1.4) in the Stratonovich sense, in which case one would have $U^{\text{Itô}} \stackrel{\text{def}}{=} U_{-1/2}$ and $U^{\text{geo}} \stackrel{\text{def}}{=} U_0$.

Since the general theory of regularity structures is completely agnostic to the specific structure of our problem, this begs the question whether there are solution theories for (1.2) that are equivariant under changes of coordinates or that satisfy Itô's isometry in the above sense. The goal of this article is to show the following

1. Among all natural solution theories for (1.2) there is a 15-dimensional family, parametrised by an affine subspace of $\mathcal{S}$, which are all equivariant under changes of coordinates.
2. There is a 19-dimensional family of solution theories, parametrised by an affine subspace of $\mathcal{S}$, which all satisfy Itô's isometry.
3. There is a two-dimensional family of solution theories that are equivariant under changes of coordinates and satisfy Itô's isometry *simultaneously*. (See Remark 1.13 and Proposition 3.15.)
4. There is a one-dimensional family $(U^{\mathfrak{b}})_{\mathfrak{b}\in\mathbf{R}}$ of solution theories (let us call it the 'canonical family', see (1.10)) which are furthermore obtained as limits to equations of the type
$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\,\xi_i^\varepsilon + H_\varepsilon^\alpha(u)\;,$$
where the counterterm H_ε is guaranteed to satisfy $H_\varepsilon(u) = 0$ whenever $u \in \mathbf{R}^d$ is such that $\Gamma(u) = 0$ and $D\sigma_i(u) = 0$.
5. In the particular cases when Γ determines the Levi-Civita connection or when the curvature tensor determined by Γ vanishes, all solution theories in the canonical family do coincide.
6. In all special cases where a canonical notion of solution was previously known to exist, all solution theories in the canonical family do coincide with the previously constructed solution. This includes in particular the KPZ equation (in which case we recover the Hopf-Cole solution) and the case $\Gamma = 0$ in which case we recover the usual Itô solutions.

We insist again on the fact that in the classical case of finite-dimensional SDEs there does *not* exist any natural notion of solution which is equivariant under changes of coordinates and satisfies Itô's isometry simultaneously. The closest we can come to this in the case of SDEs would be to define $U(\sigma, h)$ as the solution to

$$dx^\alpha = h^\alpha(x)\,dt + \sigma_i^\alpha(x)\,dW_i(t) - \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(x)\,g^{\beta\gamma}(x)\,dt\;,$$
$$dx = h(x)\,dt + \sigma_i(x)\circ dW_i(t) - \frac{1}{2}\nabla_{\sigma_i}\sigma_i\,dt\;,$$

where Γ are the Christoffel symbols for the (inverse) metric $g^{\beta\gamma} = \sigma_i^\alpha\sigma_i^\beta$. Indeed, it is a simple exercise to show that these two equations yield the same process. Furthermore, it follows immediately from the first expression that the law of this process only depends on g and h (rather than σ and h), while it follows from the second expression that it is independent of the coordinate system used to write the equation. However, there does not appear to exist any notion of stochastic integration $\star$ which is defined on a natural class of integrands and such that the process defined above solves $dx = h(x)\,dt + \sigma_i(x)\star dW_i(t)$.

1.2 Formulation of main result

We now introduce some notation allowing us to formulate our main results. We fix some ambient space $\mathbf{R}^d$ (even in the case where $\mathscr{M}$ is an arbitrary compact manifold we view it as a submanifold of $\mathbf{R}^d$ by Nash embedding) and we consider arbitrary smooth functions $\Gamma^\alpha_{\beta\gamma}, \sigma_i^\alpha : \mathbf{R}^d \to \mathbf{R}$ as above, the only constraint being that $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. Here and subsequently, Greek indices run over $\{1,\dots,d\}$ while Roman indices run over $\{1,\dots,m\}$, with m being the number of driving noises. We also fix a collection of smooth functions h^α and K^α_β and, for some arbitrary but fixed $a \in (0,\frac{1}{2})$, we denote by $\mathscr{C}^a_\star$ a suitable space of parabolic a-Hölder continuous functions $u\colon \mathbf{R}_+ \times S^1 \to \mathbf{R}^d$ with possible blow-up at finite time. (See Section 2 below for a precise definition.) For $\varepsilon > 0$ and ϱ a space-time mollifier (compactly supported, integrating to 1, and such that $\varrho(t,-x) = \varrho(t,x)$), we then denote by $U^{\mathrm{geo}}_\varepsilon(\Gamma, K, \sigma, h)$ the map from $\mathscr{C}^a$ to the space of probability measures on $\mathscr{C}^a_\star$ assigning to $u_0 \in \mathscr{C}^a$ the law of the maximal solution to

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + K^\alpha_\beta(u)\,\partial_x u^\beta + h^\alpha(u) + \sigma_i^\alpha(u)\,\xi_i^{(\varepsilon)}\;, \quad (1.5)$$

where $u : \mathbf{R}_+ \times S^1 \to \mathbf{R}^d$, $\xi_i^{(\varepsilon)} = \varrho_\varepsilon * \xi_i$ and $\varrho_\varepsilon(t,x) = \varepsilon^{-3}\varrho(t/\varepsilon^2, x/\varepsilon)$. We henceforth denote by $\mathscr{B}^a_\star$ the space of continuous maps from $\mathscr{C}^a(S^1,\mathbf{R}^d)$ into the space of probability measures on $\mathscr{C}^a_\star$.

In order to formulate our results, we first note that the class of equations of the type (1.5) is invariant under composition by diffeomorphisms in the following way. We interpret Γ as the Christoffel symbols for an arbitrary connection on $\mathbf{R}^d$ and, for each i, the $(\sigma_i^\alpha)_\alpha$ as the components of a vector field on $\mathbf{R}^d$. Given a diffeomorphism φ of $\mathbf{R}^d$, we then act on connections Γ, vector fields σ and $(1,1)$-tensors K in the usual way by imposing that

$$(\varphi \bullet \Gamma)^\alpha_{\eta\zeta}(\varphi(u))\,\partial_\beta\varphi^\eta(u)\,\partial_\gamma\varphi^\zeta(u) = \partial_\mu\varphi^\alpha(u)\,\Gamma^\mu_{\beta\gamma}(u) - \partial^2_{\beta\gamma}\varphi^\alpha(u)\;, \quad (1.6a)$$
$$(\varphi \bullet \sigma)^\alpha(\varphi(u)) = \partial_\beta\varphi^\alpha(u)\,\sigma^\beta(u)\;, \quad (1.6b)$$

$$(\varphi \bullet K)^\alpha_\eta(\varphi(u))\,\partial_\beta \varphi^\eta(u) = \partial_\mu \varphi^\alpha(u)\,K^\mu_\beta(u)\;. \tag{1.6c}$$

(One can verify that both of these do indeed describe left actions of the group of diffeomorphisms.) Similarly, given a map $U\colon u_0 \mapsto u$, we write $\varphi \bullet U$ for the map that maps $\varphi \circ u_0$ to $\varphi \circ u$. With these notations, a simple calculation shows that one has the following equivariance property of $U^{\mathrm{geo}}_\varepsilon$ under the action of the diffeomorphism group:

$$\varphi \bullet U^{\mathrm{geo}}_\varepsilon(\Gamma, K, \sigma, h) = U^{\mathrm{geo}}_\varepsilon(\varphi \bullet \Gamma, \varphi \bullet K, \varphi \bullet \sigma, \varphi \bullet h)\;.$$

Recall that the covariant derivative $\nabla_X Y$ of a vector field Y in the direction of another vector field X is the vector field given by

$$(\nabla_X Y)^\alpha(u) = X^\beta(u)\,\partial_\beta Y^\alpha(u) + \Gamma^\alpha_{\beta\gamma}(u)\,X^\beta(u)\,Y^\gamma(u)\;. \tag{1.7}$$

It is straightforward to verify that this definition satisfies

$$\varphi \bullet (\nabla_X Y) = (\varphi \bullet \nabla)_{\varphi \bullet X}(\varphi \bullet Y)\;,$$

where $\varphi \bullet \nabla$ denotes the covariant differentiation built as in (1.7), but with Γ replaced by $\varphi \bullet \Gamma$.

This allows us to build a number of different vector fields from Γ and σ, like for example $\sum_i \nabla_{\sigma_i}\sigma_i$, which we simply write as $\nabla_\circ \circ$ (each circle denotes an instance of σ_i, with different values of the index corresponding to different colours and all indices being summed over). With this notation at hand, consider the following collection of 14 triple covariant derivatives:

$$\begin{aligned}\mathfrak{V} = \{&\nabla_\circ \nabla_\bullet \nabla_\bullet \circ,\ \nabla_\bullet \nabla_\bullet \nabla_\circ \circ,\ \nabla_\bullet \nabla_{\nabla_\bullet \circ} \circ,\ \nabla_{\nabla_\bullet \circ} \nabla_\bullet \circ,\ \nabla_{\nabla_\circ \bullet} \nabla_\bullet \circ,\ \nabla_{\nabla_\bullet \bullet} \nabla_\circ \circ,\ \nabla_{\nabla_\bullet \nabla_\bullet \circ} \circ,\\ &\nabla_{\nabla_\bullet \nabla_\circ \circ} \bullet,\ \nabla_{\nabla_{\nabla_\bullet \circ} \circ} \bullet,\ \nabla_{\nabla_{\nabla_\bullet \circ} \bullet} \circ,\ \nabla_\circ \nabla_{\nabla_\bullet \circ} \bullet,\ \nabla_{\nabla_\circ \nabla_\bullet \circ} \bullet,\ \nabla_{\nabla_{\nabla_\circ \circ} \bullet} \bullet,\ \nabla_\bullet \nabla_{\nabla_\circ \circ} \bullet\ \}.\end{aligned} \tag{1.8}$$

(The only element missing from the list is $\nabla_\bullet \nabla_\circ \nabla_\bullet \circ$, the reason being that it can be written as a linear combination of the 14 other terms, see (6.24) below.) It will be convenient to view $\mathfrak{V}$ as an 'abstract' set of symbols, to write $\mathcal{V} = \langle \mathfrak{V} \rangle$ for the vector space that it generates[1], and to write $\Upsilon_{\Gamma,\sigma}\colon \mathcal{V} \to \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$ for the map that turns each symbol into the corresponding vector field on $\mathbf{R}^d$, so that for example $\Upsilon_{\Gamma,\sigma}\nabla_\bullet \nabla_{\nabla_\bullet \circ} \circ = \sum_{i,j} \nabla_{\sigma_i} \nabla_{\nabla_{\sigma_i}\sigma_j} \sigma_j$.

A special case of (1.5) is given by the case $\Gamma = 0$ and σ constant, i.e. $D\sigma = 0$, in which case it reduces to the heat equation with additive noise which obviously requires no renormalisation. It is then natural to expect that if (1.5) is sufficiently "close to" the additive stochastic heat equation near some fixed point $u_0 \in \mathbf{R}^d$, then the counterterms required to give meaning to its solutions vanish at u_0. This motivates the introduction of the space $\mathcal{V}^{\mathrm{nice}} \subset \mathcal{V}$ consisting of those elements $\tau \in \mathcal{V}$ such that, for all choices of Γ and σ, if u_0 is a point such that $\Gamma(u_0) = 0$ and

[1] We will stick as much as possible to the convention that the vector space generated by a set denoted by a Gothic symbol is denoted by the corresponding calligraphic symbol.

$D\sigma(u_0) = 0$, then $(\Upsilon_{\Gamma,\sigma}\tau)(u_0) = 0$. We will see in Remark 3.17 below that $\mathscr{V}^{\text{nice}}$ is a 12-dimensional subspace of $\mathscr{V}$.

We write $H_{\Gamma,\sigma}$ for the vector field

$$H^\alpha_{\Gamma,\sigma}(u) = R^\alpha_{\beta\gamma\eta}(u)\,\sigma^\beta_i(u)\big(\nabla_{\sigma_j}\sigma_i - 2\nabla_{\sigma_i}\sigma_j\big)^\gamma(u)\,\sigma^\eta_j(u)$$
$$= -R^\alpha_{\beta\gamma\eta}(u)\,g^{\beta\zeta}(u)\,(\nabla_\zeta g)^{\gamma\eta}(u)\;, \tag{1.9}$$

see Lemma 3.14, where summation over i, j is implicit, g is given by (1.3), and the Riemannian curvature tensor R is defined from Γ as usual through the identity

$$R^\alpha_{\beta\gamma\eta}X^\gamma Y^\eta Z^\beta = (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)^\alpha$$

for any three vector fields X, Y, Z and ∇ defined by (1.7). The vector field H can be written as a linear combination of elements of $\mathfrak{V}$ and it turns out that this linear combination belongs to $\mathscr{V}^{\text{nice}}$, so that it determines a one-dimensional linear subspace $\mathscr{V}_\star \subset \mathscr{V}^{\text{nice}}$ spanned by the element $\tau_\star \in \mathscr{V}^{\text{nice}}$ such that $H_{\Gamma,\sigma} = \Upsilon_{\Gamma,\sigma}\tau_\star$. We also choose an arbitrary complement $\mathscr{V}_\star^\perp$ so that $\mathscr{V}^{\text{nice}} = \mathscr{V}_\star \oplus \mathscr{V}_\star^\perp$. Incidentally, and this is the reason why this subspace is important, we will show in Proposition 3.15 below that $\mathscr{V}_\star$ equals all of those elements $\tau \in \mathscr{V}^{\text{nice}}$ such that $\Upsilon_{\Gamma,\sigma}\tau$ is a vector field that furthermore only depends on the σ_i through the inverse metric g defined by (1.3).

Having introduced all of these auxiliary vector fields, our main result can be stated as follows (parts of this result were already announced in [Hai16]). Throughout this article, we consider $a \in (0, \frac{1}{2})$ to be a fixed Hölder exponent. We also write Moll for the set of all compactly supported functions $\varrho\colon \mathbf{R}^2 \to \mathbf{R}$ integrating to 1, such that $\varrho(t, -x) = \varrho(t, x)$, and such that $\varrho(t, x) = 0$ for $t \le 0$ (i.e. ϱ is non-anticipative). Furthermore, we call "smooth data" a choice of dimensions d and m, as well as a choice of smooth functions Γ, K, h and σ as above.

Theorem 1.2 *For every mollifier $\varrho \in$ Moll there exists a unique choice of constants $\bar c \in \mathbf{R}_+$ and $c \in \mathscr{V}_\star^\perp$, as well as a choice of $\hat c \in \mathbf{R}$, such that the following statements are true.*

1. *For every $\mathfrak{b} \in \mathbf{R}$ and every smooth data, one has*

$$\lim_{\varepsilon\to0} U^{\text{geo}}_\varepsilon\Big(\Gamma, K, \sigma, h - \frac{\bar c}{\varepsilon}\nabla_{\sigma_i}\sigma_i + H_{\Gamma,\sigma}\Big(\mathfrak{b} + \hat c + \frac{\log\varepsilon}{4\sqrt{3}\pi}\Big) + \Upsilon_{\Gamma,\sigma}c\Big)$$
$$= U^{\mathfrak{b}}(\Gamma, K, \sigma, h)\;, \tag{1.10}$$

   in $\mathscr{B}^a_\star$, for some limit $U^{\mathfrak{b}}$ independent of ϱ.
2. *For every $\mathfrak{b} \in \mathbf{R}$ and every smooth data, the process defined by $U^{\mathfrak{b}}(\Gamma, \sigma, h)$ is a Feller Markov process on $\mathscr{C}^a(S^1, \mathbf{R}^d)$ with possibly finite explosion time.*
3. *For every $\mathfrak{b} \in \mathbf{R}$ and every diffeomorphism φ of $\mathbf{R}^d$, one has the change of variables formula*

$$\varphi \bullet U^{\mathfrak{b}}(\Gamma, K, \sigma, h) = U^{\mathfrak{b}}(\varphi \bullet \Gamma, \varphi \bullet K, \varphi \bullet \sigma, \varphi \bullet h)\;, \tag{1.11}$$

   valid for all smooth data.

4. *For every* $\mathfrak{b} \in \mathbf{R}$ *and every smooth data, one has*

$$U^{\mathfrak{b}}(\Gamma, K, \sigma, h) = U^{\mathfrak{b}}(\Gamma, K, \bar{\sigma}, h)\,, \tag{1.12}$$

for every $\bar{\sigma}$ *such that* $\bar{\sigma}_i^\alpha \bar{\sigma}_i^\beta = \sigma_i^\alpha \sigma_i^\beta$. *(Implicit summation over i.)*

5. *In the special case* $\Gamma = 0$ *and* $K = 0$, $U^{\mathfrak{b}}(0, 0, \sigma, h)$ *coincides with the mild solution to the system of stochastic PDEs*

$$du^\alpha = \partial_x^2 u^\alpha \, dt + h^\alpha(u)\, dt + \sigma_i^\alpha(u)\, dW_i\,, \tag{1.13}$$

where the W_i *are independent* $L^2(S^1)$*-cylindrical Wiener processes.*

6. *In the special case* $\Gamma = 0$, $d = m$, *and* σ *a constant multiple of the identity matrix,* $U^{\mathfrak{b}}(0, K, \sigma, h)$ *coincides with the maximal solution constructed in [Hai11].*

The case of a Riemannian manifold $\mathscr{M}$ is so important that we state it as a separate result. Given a collection $\{\sigma_i\}_{i=1}^m$ on a Riemannian manifold $(\mathscr{M}, g)$, we define $\Upsilon_\sigma \colon \mathfrak{V} \to \Gamma^\infty(T\mathscr{M})$ as before as the map from $\mathfrak{V}$ into the set of smooth vector fields on $\mathscr{M}$ assigning a symbols to the corresponding triple covariant derivative of the σ's. The slight difference is that this time we view these as vector fields on $\mathscr{M}$ rather than $\mathbf{R}^n$ and we use the convention that the covariant differentiation is the one given by the Levi-Civita connection on $\mathscr{M}$. Writing $\Gamma_\sigma^\infty(T\mathscr{M}) = \operatorname{range} \Upsilon_\sigma$, the following result shows that in the natural geometric context the collection $\mathfrak{V}$ has quite a lot of redundancies.

Lemma 1.3 *Let* $(\mathscr{M}, g)$ *be a Riemannian manifold and let* $\{\sigma_i\}_{i=1}^m$ *be a collection of smooth vector fields such that the tensor* $\sum_{i=1}^m (\sigma_i \otimes \sigma_i)$ *equals the inverse of the metric tensor and such that furthermore* $\sum_{i=1}^m \nabla_{\sigma_i} \sigma_i = 0$.

Then, the space $\Gamma_\sigma^\infty(T\mathscr{M})$ *is of dimension at most* 8. *Furthermore, one has* $\bigcap_\sigma \Gamma_\sigma^\infty(T\mathscr{M}) = \langle \nabla R \rangle$, *where* R *denotes the scalar curvature and the intersection ranges over all choices of* σ *with the above properties.*

Proof. It suffices to note that since $\sum_{i=1}^m \nabla_{\sigma_i} \sigma_i = 0$, one has $\Upsilon_\sigma \tau = 0$ for every one of the 5 symbols in $\mathfrak{V}$ that contain $\nabla_{\circ}\circ$ as a subsymbol. To eliminate one more degree of freedom, it suffices to note that since $\nabla g = 0$ by definition of the Levi-Civita connection, one has $\Upsilon_\sigma \tau_\star = 0$ by (1.9). □

Remark 1.4 One may wonder whether such collections of vector fields do exist. This is always the case since it suffices to consider a smooth isometric embedding of $\mathscr{M}$ into $\mathbf{R}^d$ and to choose for $\sigma_i(p)$ the orthogonal projection of the ith canonical basis vector onto $T_p\mathscr{M}$, see for example [Hsu02] or Lemma 4.9 below.

Theorem 1.5 *Let* $\mathscr{M}$, g *and* σ *be as in Lemma 1.3, let* $h \in \Gamma^\infty(T\mathscr{M})$, *and let* $\varrho \in \mathrm{Moll}$. *For any* $V_{\varrho,\sigma} \in \Gamma_\sigma^\infty(T\mathscr{M})$ *and any* $\varepsilon \in (0, 1]$, *we denote by* u_ε *the (local) solution to the random PDE*

$$\partial_t u_\varepsilon = \nabla_{\partial_x u_\varepsilon} \partial_x u_\varepsilon + h(u_\varepsilon) + \sum_{i=1}^m \sigma_i(u_\varepsilon)\, \xi_i^\varepsilon + V_{\varrho,\sigma}(u_\varepsilon)\,. \tag{1.14}$$

Then, u_ε converges in probability (up to a possible explosion time) to a limit u which may in general depend on ϱ, σ and $V_{\varrho,\sigma}$.

However, there exists a unique choice $(\varrho,\sigma) \mapsto V_{\varrho,\sigma}^{\text{canon}}$ (corresponding to the 'canonical family' of the previous theorem) such that

1. *The limit u is independent of ϱ and σ.*
2. *For every choice of $\mathscr{M}$, g and σ, whenever $p \in \mathscr{M}$ is such that $(\nabla_{\sigma_i}\sigma_j)(p) = 0$ for all i and j, one has $V_{\varrho,\sigma}^{\text{canon}}(p) = 0$.*
3. *One can find $\varrho \mapsto c_\varrho \in \mathscr{V}$ such that $V_{\varrho,\sigma}^{\text{canon}} = \Upsilon_\sigma c_\varrho$.*

Finally, every choice $(\varrho,\sigma) \mapsto V_{\varrho,\sigma}$ satisfying condition 1 is of the form $V_{\varrho,\sigma} = V_{\varrho,\sigma}^{\text{canon}} + c\nabla R$.

Remark 1.6 It is natural to call the above limit with the choice $V_{\varrho,\sigma} = V_{\varrho,\sigma}^{\text{canon}}$ "the" solution to the SPDE

$$\partial_t u = \nabla_{\partial_x u}\partial_x u + h(u) + \sum_{i=1}^m \sigma_i(u)\,\xi_i \;. \tag{1.15}$$

We do however conjecture that the invariant measure of this process when $h = 0$ is *not* the Brownian loop measure (at half its natural speed) on $\mathscr{M}$. Instead, that measure should be invariant for this process with $h = \frac{1}{32}\nabla R$, see Conjecture 4.5 below for more details and a semi-heuristic justification of this claim. (The reason for the value $\frac{1}{32}$ rather than the value $\frac{1}{8}$ appearing there is the absence of the factor $\sqrt{2}$ in front of the noise term in (1.15), combined with the fact that ∇R is 4-linear in σ.)

Remark 1.7 In the particular case of the KPZ equation, the combination of properties 3 and 5 shows that our specific choice of the finite constant c singles out the Hopf-Cole solution. Furthermore, since the intrinsic curvature of a one-dimensional Riemannian manifold always vanishes, one has $H_{\Gamma,\sigma} = 0$ in this case, which explains the cancellation of the two logarithmically divergent constants in [Hai13].

Remark 1.8 In the case of the generalised KPZ equation in $\mathbf{R}$, i.e. for $d = 1$, given by

$$\partial_t u = \partial_x^2 u + \Gamma(u)(\partial_x u)^2 + K(u)\,\partial_x u + h(u) + \sigma(u)\,\xi \;,$$

one has $H_{\Gamma,\sigma} = 0$ as in the previous remark, which gives the cancellation of many logarithmically divergent constants. Furthermore, property 3 gives the chain rule for this equation.

Remark 1.9 We will actually show a slightly stronger version of (1.11), namely that for any two open sets $U, V \subset \mathbf{R}^d$ and any diffeomorphism $\varphi\colon U \to V$, (1.11) also holds if we restrict ourselves to solutions with initial conditions taking values in U (respect. V), stopped whenever they exit these domains. The reason why this is slightly stronger than (1.11) is that there are diffeomorphisms $U \to V$ that cannot be extended to a global diffeomorphism on $\mathbf{R}^d$ due to topological obstructions.

Remark 1.10 The only part which is non-canonical is the value of $\hat{c}$. However, two different choices for this component simply correspond to a reparametrisation of the family $\mathfrak{b} \mapsto U^{\mathfrak{b}}$ which is itself perfectly canonical. Furthermore, as an immediate consequence of (1.10), the solution theories $U^{\mathfrak{b}}$ for different values of $\mathfrak{b}$ are related by $U^{\bar{\mathfrak{b}}}(\Gamma, K, \sigma, h) = U^{\mathfrak{b}}(\Gamma, K, \sigma, h + (\bar{\mathfrak{b}} - \mathfrak{b})H_{\Gamma,\sigma})$.

Remark 1.11 As already mentioned earlier, part 4 of this statement is a version of Itô's isometry in this context (which is consistent with part 5) while part 3 corresponds to the classical change of variables formula. In this sense, each one of the solution theories $U^{\mathfrak{b}}$ for (1.2) behaves simultaneously like both an 'Itô solution' and a 'Stratonovich solution'.

Remark 1.12 It is important that one chooses the mollifier ϱ in such a way that $\varrho(t, x) = \varrho(t, -x)$, otherwise one may have to subtract an additional term of the form $f^\alpha_\beta(u)\partial_x u^\beta$ for suitable f in order to get the same limit and to restore in the limit the $x \leftrightarrow -x$ symmetry which is then broken by the approximation.

Remark 1.13 If, instead of working with counterterms belonging to $\mathcal{V}^{\text{nice}}$, we had chosen to work with counterterms in the "full" space $\mathcal{V}$, the degree of freedom $\mathfrak{b}$ appearing in the theorem would turn out to be two-dimensional instead of one-dimensional. The second "free" direction is then given by a counterterm proportional to

$$\hat{H}^\alpha(u) \stackrel{\text{def}}{=} (\nabla_\zeta R^\alpha_{\beta\gamma\eta})(u) g^{\zeta\gamma}(u) g^{\beta\eta}(u) , \tag{1.16}$$

see Proposition 3.15 and Lemma 3.14.

In the case when Γ is the Levi-Civita connection given by the (inverse) Riemannian metric g determined by σ through (1.3), a straightforward calculation shows that (1.16) is equivalent to the simpler identity

$$\hat{H}^\alpha = \frac{1}{2}\nabla^\alpha R(u) ,$$

where R denotes the scalar curvature.

This suggests that the corresponding family of stochastic processes has invariant measures given by the Brownian loop measure, weighted by the (exponential of) the integral of the scalar curvature along the loop. Since this is the only degree of freedom in the renormalisation group associated to our equation that isn't determined by symmetry considerations, it also suggests that different elements from this family of measures can arise from rather natural-looking approximations to the Brownian loop measure. This is a well-known fact that has already been pointed out in the physics literature of the early seventies [Che72, Um74] and appears in more recent mathematical works on the topic [Dar84, AD99]. In our setting, it appears very natural to restrict our renormalisation to $\mathcal{V}^{\text{nice}}$, which fixes an origin for this degree of freedom, albeit in a rather arbitrary way. Another natural choice would be to choose it in such a way that the Brownian loop measure is indeed invariant for (1.2). It is not clear at this stage whether these two choices of origin coincide. We will revisit this point in more detail in Section 4.3 where we conjecture that they do *not* and that these two natural choices of origin differ by $\frac{1}{8}$.

1.3 Structure of the article

Our strategy for the proof for Theorem 1.2, which takes up the remainder of this article, goes as follows. After recalling some of the concepts and notations from the theory of regularity structures in Section 2.1 and 2.2, we apply the results of [BHZ18, CH16, BCCH17] to our problem in Sections 2.3 and 2.4. This shows that there exists a finite-dimensional space $\mathcal{S}$ of formal expressions (see Section 2.4 for a precise definition of $\mathcal{S}$) with a canonical embedding $\mathscr{V} \subset \mathcal{S}$, together with natural valuations $\Upsilon_{\Gamma,\sigma} \colon \mathcal{S} \to \mathscr{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$, as well as a choice of renormalisation constants $C^{\text{BPHZ}}_{\varepsilon,\text{geo}} \in \mathcal{S}$ such that $U^{\text{geo}}_\varepsilon(\Gamma, K, \sigma, h + \Upsilon_{\Gamma,\sigma} C^{\text{BPHZ}}_{\varepsilon,\text{geo}})$ converges as $\varepsilon \to 0$ to some limit $\bar{U}$ independent of ϱ. We similarly show that, for a different approximation $U^{\text{Itô}}_\varepsilon$ to (1.2) (instead of only hitting the noise with the mollifier, we hit the whole right hand side of the equation with it), one has an analogous result with a different set of renormalisation constants $C^{\text{BPHZ}}_{\varepsilon,\text{Itô}}$. The point here is that we know a priori from [CH16] that, with this specific choice of renormalisation constants. the two limits obtained from these two different approximations do coincide, call it U^{BPHZ}.

In Section 3.1, we then introduce two subspaces $\mathcal{S}_{\text{Itô}} \subset \mathcal{S}$ and $\mathcal{S}_{\text{geo}} \subset \mathcal{S}$ with the property that for any fixed ε, $U^{\text{geo}}_\varepsilon(\Gamma, K, \sigma, h + \Upsilon_{\Gamma,\sigma}\tau)$ satisfies Property 3 of Theorem 1.2 for any $\tau \in \mathcal{S}_{\text{geo}}$, while $U^{\text{Itô}}_\varepsilon(\Gamma, K, \sigma, h + \Upsilon_{\Gamma,\sigma}\tau)$ satisfies Property 4 of Theorem 1.2 for any $\tau \in \mathcal{S}_{\text{Itô}}$. In Sections 3.3 and 3.4 we then show that the renormalisation constants $C^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ (resp. $C^{\text{BPHZ}}_{\varepsilon,\text{Itô}}$) 'almost' belong to $\mathcal{S}_{\text{geo}}$ (resp. $\mathcal{S}_{\text{Itô}}$) in the sense that there exist $\hat{C}^{\text{BPHZ}}_{\varepsilon,\text{geo}} \in \mathcal{S}_{\text{geo}}$ such that $C^{\text{BPHZ}}_{\varepsilon,\text{geo}} - \hat{C}^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ converges to a finite limit in $\mathcal{S}$, and similarly for $C^{\text{BPHZ}}_{\varepsilon,\text{Itô}}$. This relies in a crucial way on the fact that the corresponding two approximation procedures for our equation already exhibit the required symmetries at fixed $\varepsilon > 0$. In order to transfer a convergence result at the level of stochastic processes to the corresponding result at the level of renormalisation constants, we also rely on the injectivity of the map $h \mapsto U^{\text{BPHZ}}(\Gamma, K, \sigma, h)$, which is shown in Section 3.2.

In Section 3.5, we combine both properties to show that it is possible to find a solution theory that exhibits both Properties 3 and 4 simultaneously. This relies on the definition of a space $\mathcal{S}_{\text{both}} \subset \mathcal{S}$ of elements τ such that, given any two σ, $\bar{\sigma}$ such that $\bar{\sigma}^\alpha_i \bar{\sigma}^\beta_i = \sigma^\alpha_i \sigma^\beta_i$, the term $(\Upsilon_{\Gamma,\sigma} - \Upsilon_{\Gamma,\bar{\sigma}})\tau$ transforms like a vector field. The crucial remark which underpins our argument is the fact that one has the identity

$$\mathcal{S}_{\text{both}} = \mathcal{S}_{\text{Itô}} + \mathcal{S}_{\text{geo}} . \tag{1.17}$$

The intersection of these two spaces furthermore consists of the subspace generated by $\tau_\star$ as well as the element generating the counterterm (1.16). We eliminate the latter by restricting ourselves to the subspace $\mathcal{S}^{\text{nice}} \subset \mathcal{S}$ consisting of those terms that vanish at points where both Γ and $\partial\sigma$ vanish. It turns out that we can do this thanks to the fact that the BPHZ counterterms $C^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ and $C^{\text{BPHZ}}_{\varepsilon,\text{Itô}}$ both belong to $\mathcal{S}^{\text{nice}}$ for any choice of mollifier. We then show that (1.17) still holds if each of the spaces appearing there is replaced by its intersection with $\mathcal{S}^{\mathrm{nice}}$, but their intersection now consists solely of multiples of $\tau_\star$.

Remark 1.14 As already mentioned, the analogous identity to (1.17) does not hold in the case of SDEs. Indeed, in that case, $\mathcal{S}$ consists of all linear combinations of the terms $\{\sigma_i^\beta \partial_\beta \sigma_i^\alpha\}$ (α is a free index and summation over β and i is implied). Since this term is not invariant under changes of coordinates and cannot be written in terms of g alone, one has $\mathcal{S}_{\mathrm{Itô}} = \mathcal{S}_{\mathrm{geo}} = 0$. On the other hand, this term does belong to $\mathcal{S}_{\mathrm{both}}$ since, whenever $\bar\sigma_i^\alpha \bar\sigma_i^\beta = \sigma_i^\alpha \sigma_i^\beta$ (summation over i implied),

$$\sigma_i^\alpha \,\partial_\alpha \sigma_i^\beta - \bar\sigma_i^\alpha \,\partial_\alpha \bar\sigma_i^\beta = \nabla_{\sigma_i}\sigma_i - \nabla_{\bar\sigma_i}\bar\sigma_i$$

for any choice of connection and it therefore does transform like a vector field.

In Section 4.1 we then combine various existing results with a simple explicit calculation to show that the divergent part $\hat C^{\mathrm{BPHZ}}_{\varepsilon,\mathrm{geo}}$ is indeed of the form shown in part 1 of Theorem 1.2, which in particular relies on the fact that $\mathcal{S}_{\mathrm{Itô}} \cap \mathcal{S}_{\mathrm{geo}} \cap \mathcal{S}^{\mathrm{nice}}$ coincides with $\mathcal{V}_\star$. In order to show that the process constructed in this way coincides with the classical Itô solution in the case $\Gamma = K = 0$, we show that in this particular case the counterterms for the 'Itô approximation' vanish, i.e. $U^{\mathfrak{b}}(0,0,h,\sigma) = \lim_{\varepsilon\to 0} U_\varepsilon^{\mathrm{Itô}}(0,0,h,\sigma)$ without any renormalisation needed. This is a consequence of the fact that $\mathcal{S}_{\mathrm{Itô}}$ has the property that $\Upsilon_{0,\sigma}\tau = 0$ for every $\tau \in \mathcal{S}_{\mathrm{Itô}}$, see Proposition 6.11 below. The fact that $U_\varepsilon^{\mathrm{Itô}}(0,0,h,\sigma)$ converges to the classical Itô solution is easy to show. Section 4.4 is then devoted to a discussion of the natural geometric situation mentioned as a motivation at the beginning of the introduction and contains the proof of Theorem 1.5.

Sections 5 and 6 are devoted to the proof of (1.17). While the inclusion $\mathcal{S}_{\mathrm{Itô}} + \mathcal{S}_{\mathrm{geo}} \subset \mathcal{S}_{\mathrm{both}}$ is trivial, its converse is not, as already noted in Remark 1.14. In order to show this, we need a better understanding of the structure of the space $\mathcal{S}$. This is given by the notion of a "T-algebra" which we introduce in Section 5. In a nutshell, this notion encapsulates the main features of the spaces $\mathcal{C}^\infty(V,(V^*)^{\otimes k}\otimes V^{\otimes \ell})$: they come with an action of two copies of the symmetric group (allowing to permute the factors V and V^* in the target space), a product, a notion of derivative (identifying $L(V,(V^*)^{\otimes k}\otimes V^{\otimes \ell})$ with $(V^*)^{\otimes (k+1)}\otimes V^{\otimes \ell}$), as well as a partial trace (pair the last factor V^* with the last factor V). Our main abstract results are a characterisation of the 'free T-algebra' obtained from a collection of generators, see Theorem 5.16 in Section 5.2, as well as a non-degeneracy result for the morphisms from the free T-algebra into the spaces $\mathcal{C}^\infty(V,(V^*)^{\otimes k}\otimes V^{\otimes \ell})$, see Theorem 5.22 in Section 5.3.

This provides us with a rigorous underpinning for a diagrammatic calculus on the possible counterterms appearing in the renormalisation of (1.2), as well as the tools required to give a clean characterisation of the spaces $\mathcal{S}_{\mathrm{geo}}$ in Section 3.3 and $\mathcal{S}_{\mathrm{Itô}}$ in Section 6.2. The actual proof of (1.17) is then given in Section 6.3. Unfortunately, it is not as elegant as one may wish and relies on a dimension counting argument that appears to be somewhat *ad hoc* to the situation at hand. It does however rely strongly on the identifications of $\mathcal{S}_{\mathrm{geo}}$ and $\mathcal{S}_{\mathrm{Itô}}$ which appear to have a more universal flavour.

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2 The BPHZ solution

In this section, we recall how the results of [BHZ18, CH16, BCCH17] can be combined to produce a number of natural candidate "solution theories" for the class of SPDEs (1.2) on $\mathbf{R}^d$.

2.1 Construction of the regularity structure

Recall first that a class of stochastic PDEs is naturally associated to a "rule" (in the technical sense of [BHZ18]) defining a corresponding regularity structure. Such a rule describes how the different noises and convolution operators combine via the non-linearity to form higher-order stochastic processes that provide a good local description of the solution which is stable under suitable limits.

In our case, there is only one convolution operator (convolution against the heat kernel, or rather a possibly mollified truncation thereof), while there are m noises, all having the same regularity. This motivates the introduction of a label set $\mathfrak{L} = \{|, \circ_1, \ldots, \circ_m\}$ with corresponding degrees $\deg(|) = 2 - \kappa$ and $\deg(\circ_i) = -\frac{3}{2} - \kappa$ for $\kappa > 0$ sufficiently small ($\kappa < 1/100$ will do). Recall that a "rule" is then a map R assigning to each element of $\mathfrak{L}$ a non-empty collection of tuples in $\mathfrak{L} \times \mathbf{N}^2$, where the element of $\mathbf{N}^2$ represents a space-time multiindex. In our case, we identify $\mathfrak{L}$ with $\mathfrak{L} \times \{0\} \subset \mathfrak{L} \times \mathbf{N}^2$ and we use the shorthand $\mathbf{|} = (|, (0,1))$ (with the "1" denoting the space direction). With these notations, the relevant rule describing the class of equations (1.2) is given by $R(\circ_i) = \{()\}$ and

$$R(|) = \{(|^k, \circ_i), (\mathbf{|}^\ell, |^k) \,:\, k \geq 0,\, \ell \in \{0,1,2\},\, i \in \{1, \ldots, m\}\}\;,$$

where we used $|^k$ to denote k repetitions of $|$ and similarly for $\mathbf{|}^\ell$.

As already remarked in [BHZ18, Sec. 5.4], the rule R is normal, subcritical and complete, so that it determines a regularity structure $(\mathcal{T}, \mathcal{G})$, together with a "renormalisation group" $\mathfrak{R}$ of extraction / contraction operations acting continuously on its space of admissible models. The space $\mathcal{T}$ is a graded vector space $\mathcal{T} = \bigoplus_\alpha \mathcal{T}_\alpha$ with each $\mathcal{T}_\alpha$ finite-dimensional. Elements of $\mathcal{T}$ are formal linear combinations of labelled trees. Here, a labelled tree $T_{\mathfrak{f}}^{\mathfrak{n}}$ consists of

- A combinatorial rooted tree T with vertex set V_T, edge set E_T and root ϱ_T.

- An edge decoration $\mathfrak{f}\colon E_T \to \mathfrak{L} \times \mathbf{N}^2$. We call the first component of $\mathfrak{f}(e)$ the 'type' of an edge e.
- A vertex decoration $\mathfrak{n}\colon V_T \to \mathbf{N}^2$.

We denote by $N_T \subset E_T$ the set of "noises", which are the edges of type $\circ_i$ for some i.

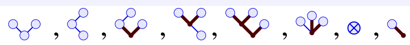
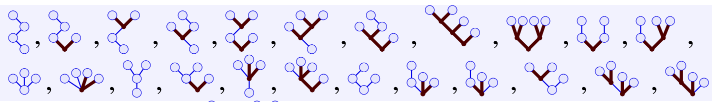
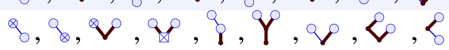
Furthermore, we restrict ourselves to trees conforming to the rule R in the following way. Given a vertex $v \in V_T$, we call the (unique) edge e_v adjacent to v and pointing towards the root ϱ_T the 'outgoing' edge and all other edges adjacent to v 'incoming' edges. The collection of incoming edges then determines a tuple $\mathcal{N}(v)$ of $\mathfrak{L} \times \mathbf{N}^2$ by collecting all of their decorations $\mathfrak{f}$. With this notation, we restrict ourselves to trees such that $\mathcal{N}(v) \in R(\mathfrak{t}_v)$, where $\mathfrak{t}_v$ is the type of the outgoing edge e_v, with the convention that $\mathfrak{t}_{\varrho_T} = |$. We furthermore, impose that $\mathfrak{n}(v) = 0$ if $\mathfrak{t}_v \in \{\circ_i\}$.

We henceforth denote by $\mathfrak{T}$ the collection of all labelled trees conforming to R. The notion of degree naturally extends to $\mathfrak{T}$ by setting

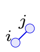
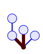
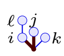
$$\deg T_{\mathfrak{f}}^{\mathfrak{n}} = \sum_{v \in V_T} |\mathfrak{n}(v)| + \sum_{e \in E_T} \deg(\mathfrak{f}(e)) ,$$

where $|(k_0, k_1)| = 2k_0 + k_1$ and $\deg(\mathfrak{t}, k) = \deg \mathfrak{t} - |k|$. With these notations, the space $\mathcal{T}$ is nothing but the vector space generated by $\mathfrak{T}$, with the grading given by the degree.

A particular role in the general theory of regularity structures is played by the trees of strictly negative degree. Denoting by $\boxtimes$ a vertex with label $(0, 1)$, by $\circ$ a vertex with label 0 and an incoming edge of type $\circ_i$ for some i and finally by $\otimes$ a vertex with label $(0, 1)$ and an incoming edge of type $\circ_i$ for some i, the complete list of elements $\mathfrak{T}_- \subset \mathfrak{T}$ of strictly negative degree is as follows, provided that $\kappa > 0$ is sufficiently small.

Degree	Elements of $\mathfrak{T}_-$
$-\frac{3}{2}^-$	
-1^-	
$-\frac{1}{2}^-$	
0^-	
0^-	

Here, we say that an element has degree α^- if it has degree $\alpha - n\kappa$ for some integer n. Recall that we have dropped for simplicity the indices from $\circ$ and $\otimes$, so that every element of this list stands for a finite collection of basis vectors, for instance

 $= \{$  $: i, j = 1, \ldots, m\},$  $= \{$  $: i, j, k, \ell = 1, \ldots, m\}.$ (2.1)

The symbols of this list playing the most important role later on are those belonging to the two lightly shaded rows, so we write $\mathfrak{S}_\circ^{(2)}$ and $\mathfrak{S}_\circ^{(4)}$ for these two collections of symbols and set $\mathfrak{S}_\circ = \mathfrak{S}_\circ^{(2)} \cup \mathfrak{S}_\circ^{(4)}$. More formally, $\mathfrak{S}_\circ^{(k)}$ consists of those labelled trees having exactly k noises that satisfy the "saturated rule"

$$R^{\mathsf{sat}}(|) = \{(|^k, \circ_i), (\mathbf{|}^2, |^k) \,:\, k \geq 0,\, i \in \{1, \ldots, m\}\}\;,$$

and for which the vertex label $\mathfrak{n}$ vanishes. We also write $\mathcal{S}_\circ$ for the real vector space generated by $\mathfrak{S}_\circ$.

Recall that the space $\mathcal{T}$ admits two "integration maps" $\mathcal{I}$ and $\mathcal{I}'$ obtained by adding to a given labelled tree a new root vertex with zero label and joining it to the old root vertex by an edge of type $|$ or $\mathbf{|}$ respectively. It also admits a "product" obtained by joining both trees at their roots and adding their root labels. Note that while $\mathcal{I}$ and $\mathcal{I}'$ are defined on all of $\mathcal{T}$, the product is only defined on some domain of $\mathcal{T} \times \mathcal{T}$. (One has for example $\cdot$ $=$ but the product of with is not defined in $\mathcal{T}$ since our rule does not allow for more than two thick edges to enter any given vertex.)

2.2 Realisations of the regularity structure and renormalisation

Fix now a decomposition $P = K + R$ of the heat kernel on the real line into a kernel K that is even (in the spatial variable), integrates to zero, and is compactly supported in a neighbourhood of the origin and a 'remainder' R that is globally smooth. Fix also a mollifier $\varrho \in \mathrm{Moll}$ with Moll as in the introduction just before Theorem 1.2. Given $\varepsilon \geq 0$ and an arbitrary m-uple $\zeta = (\zeta_i)_{i \leq m}$ of continuous functions, we then define linear maps $\mathscr{L}_\varepsilon(\zeta) \colon \mathcal{T} \to \mathscr{D}'(\mathbf{R}^2)$ by setting $(\mathscr{L}_\varepsilon(\zeta) X^k)(z) = z^k$, where X^k denotes the tree consisting of a single vertex with label $k \in \mathbf{N}^2$ and $z^k = (t,x)^k = t^{k_0} x^{k_1}$, as well as $\mathscr{L}_\varepsilon(\zeta) \circ_i = \zeta_i$. This is extended to all of $\mathcal{T}$ inductively by setting

$$\mathscr{L}_\varepsilon(\zeta)(\tau \cdot \bar\tau) = \mathscr{L}_\varepsilon(\zeta)\, \tau \cdot \mathscr{L}_\varepsilon(\zeta) \bar\tau\;, \tag{2.2}$$

as well as

$$\mathscr{L}_\varepsilon(\zeta)(\mathcal{I}\tau) = K_\varepsilon * \mathscr{L}_\varepsilon(\zeta)\, \tau\;, \quad \mathscr{L}_\varepsilon(\zeta)(\mathcal{I}'\tau) = K_\varepsilon' * \mathscr{L}_\varepsilon(\zeta)\, \tau\;,$$

where $K_\varepsilon = \varrho_\varepsilon * K$ and K_ε' is its spatial derivative. Note that if $\varepsilon > 0$, then (2.2) is well-defined also if ζ is a distribution since $\mathscr{L}_\varepsilon(\zeta)\, \tau$ is then smooth for all $\tau \in \mathfrak{T} \setminus \{\circ_i\}_i$ and the $\circ_i$'s never get multiplied between each other.

Defining the (spatially periodic) stationary stochastic processes $\xi_i^{(\varepsilon)}$ as in (1.5), we then define random linear maps $\mathbf{\Pi}_{\mathrm{geo}}^{(\varepsilon)}, \mathbf{\Pi}_{\mathrm{Itô}}^{(\varepsilon)} : \mathcal{T} \to \mathscr{D}'(\mathbf{R}^2)$ by setting

$$\mathbf{\Pi}_{\mathrm{geo}}^{(\varepsilon)} = \mathscr{L}_0(\xi^{(\varepsilon)})\;, \qquad \mathbf{\Pi}_{\mathrm{Itô}}^{(\varepsilon)} = \mathscr{L}_\varepsilon(\xi)\;. \tag{2.3}$$

Recall also from [BHZ18, Secs 5 & 6] that the free algebra $\langle\!\langle \mathfrak{T}_- \rangle\!\rangle$ generated by $\mathfrak{T}_-$ admits a natural Hopf algebra structure as well as a coaction $\Delta^- : \mathcal{T} \to \langle\!\langle \mathfrak{T}_- \rangle\!\rangle \otimes \mathcal{T}$ given by a natural "extraction-contraction" procedure. Characters of $\langle\!\langle \mathfrak{T}_- \rangle\!\rangle$ are identified with elements of $\mathcal{T}_-^*$, where $\mathcal{T}_- \subset \mathcal{T}$ is the subspace spanned by $\mathfrak{T}_-$. A simple recursion then shows the following.

Proposition 2.1 *Given K, ϱ and ε as above, there exists for $i \in \{\mathrm{geo}, \mathrm{Itô}\}$ a unique character $C^{\mathrm{BPHZ}}_{\varepsilon,i}$ of $\langle\!\langle \mathfrak{T}_- \rangle\!\rangle$ such that, setting*

$$\hat{\mathbf{\Pi}}^{(\varepsilon)}_i \tau = (C^{\mathrm{BPHZ}}_{\varepsilon,i} \otimes \mathbf{\Pi}^{(\varepsilon)}_i)\Delta^- \tau \;, \tag{2.4}$$

one has $\mathbf{E}(\hat{\mathbf{\Pi}}^{(\varepsilon)}_i \tau)(0) = 0$ for every $\tau \in \mathfrak{T}_-$.

Remark 2.2 The Hopf algebra structure of $\langle\!\langle \mathfrak{T}_- \rangle\!\rangle$ endows its space $\mathfrak{R}$ of characters with a group structure by setting

$$(f \star g)(\tau) = (f \otimes g)\Delta^- \tau \;.$$

Furthermore, given any compactly supported 2-regularising kernel K in the sense of [Hai14, Ass. 5.1], the map

$$(f, \mathbf{\Pi}) \mapsto f \star \mathbf{\Pi} \stackrel{\mathrm{def}}{=} (f \otimes \mathbf{\Pi})\Delta^- \tau \;, \tag{2.5}$$

yields a continuous action of $\mathfrak{R}$ onto the space $\mathscr{M}_K$ of models that are admissible for K.

Remark 2.3 The BPHZ character $C^{\mathrm{BPHZ}}_{\varepsilon,i}$ is always of the following form. Given any $\tau \in \mathfrak{T}$, we set $g_{\varepsilon,i}(\tau) = \mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_i \tau)(0)$ (with the convention that $\mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_{\mathrm{Itô}} \tau)(0) = 0$ if $\tau = \Xi_i \bar\tau$ for some $i \le m$ and some symbol $\bar\tau \in \mathfrak{T}$) and we extend this to an algebra morphism on $\langle\!\langle \mathfrak{T} \rangle\!\rangle$. Then, there exists a linear map $\mathscr{A}^t \colon \mathfrak{T}_- \to \langle\!\langle \mathfrak{T} \rangle\!\rangle$ (the "twisted antipode") such that $C^{\mathrm{BPHZ}}_{\varepsilon,i}(\tau) = g_{\varepsilon,i}(\mathscr{A}^t \tau)$, see [BHZ18, Eq. 6.24] where the twisted antipode is called $\hat{\mathscr{A}}$. Furthermore, the map $\mathscr{A}^t$ has the property that, viewing the element $\mathscr{A}^t \tau \in \langle\!\langle \mathfrak{T} \rangle\!\rangle$ as a linear combination of forests with trees in $\mathfrak{T}$, the noises appearing in each of these forests are in canonical one-to-one correspondence with the noises appearing in τ.

Denote now by $\mathscr{M}$ the space of all models for the regularity structure $(\mathscr{T}, \mathscr{G})$, which we turn into a Polish space by endowing it with the sequence of pseudo-metrics given in [Hai14, Sec. 3]. We also write $\mathscr{M}_\varepsilon \subset \mathscr{M}$ for the models admissible for K_ε (with $K_0 = K$) and $\mathscr{M}_\star \subset [0,1] \times \mathscr{M}$ for the closed subset consisting of those pairs $(\varepsilon, \mathbf{\Pi})$ such that $\mathbf{\Pi} \in \mathscr{M}_\varepsilon$. We naturally view $\mathscr{M}_\varepsilon$ as a subset of $\mathscr{M}_\star$ via the injection $\mathbf{\Pi} \mapsto (\varepsilon, \mathbf{\Pi})$. We know by [Hai14, BHZ18] that for any $\varepsilon > 0$, $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\mathrm{geo}}$ determines a unique random $\mathscr{M}_0$-valued random variable, which we again denote by $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\mathrm{geo}}$. The same argument also works for $(\varepsilon, \hat{\mathbf{\Pi}}^{(\varepsilon)}_{\mathrm{Itô}})$, which we view as an $\mathscr{M}_\star$-valued random variable. The main result of [CH16] can then be formulated as follows.

Theorem 2.4 *Under the above assumptions, there exists an $\mathscr{M}_0$-valued random variable $\mathbf{\Pi}^{\mathrm{BPHZ}}$ which possibly depends on the choice of decomposition $P = K + R$ but is independent of the choice of mollifier ϱ, and such that for $i \in \{\mathrm{geo}, \mathrm{Itô}\}$, $\hat{\mathbf{\Pi}}^{(\varepsilon)}_i$ converges as $\varepsilon \to 0$ to $\mathbf{\Pi}^{\mathrm{BPHZ}}$ in probability in $\mathscr{M}_\star$.*

Proof. The convergence of $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{geo}}$ to some limit $\mathbf{\Pi}^{\text{BPHZ}}$ independent of ϱ follows from [CH16, Thm. 2.31]. Let now $\delta > 0$ and write $\hat{\mathbf{\Pi}}^{(\varepsilon,\delta)}_{\text{Itô}}$ for the BPHZ renormalisation of $\mathscr{L}_\varepsilon(\xi^{(\delta)})$. For τ not containing any factor Ξ_i, it follows from the smoothness of $\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}\tau$ (at fixed $\varepsilon > 0$) that

$$\lim_{\delta\to 0} \mathbf{E}(\mathbf{\Pi}^{(\varepsilon,\delta)}_{\text{Itô}}\tau)(0) = \mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}\tau)(0) \;.$$

Furthermore, thanks to the fact that ϱ is supported on strictly negative times, one has $\mathbf{E}(\mathbf{\Pi}^{(\varepsilon,\delta)}_{\text{Itô}}\tau)(0) = 0$ for τ containing a factor Ξ_i and δ small enough, so that $\lim_{\delta\to 0}\hat{\mathbf{\Pi}}^{(\varepsilon,\delta)}_{\text{Itô}} = \hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{Itô}}$. Noting that $\|K_\varepsilon - K\|_{(2-\kappa),m}$ is bounded by some small positive power of ε and applying again [CH16, Thm. 2.31], we conclude that $\lim_{\varepsilon\to 0}\hat{\mathbf{\Pi}}^{(\varepsilon,\delta)}_{\text{Itô}} = \hat{\mathbf{\Pi}}^{(\delta)}_{\text{geo}}$ and that this convergence takes place uniformly in δ, thus concluding the proof. □

This convergence takes place in a sufficiently strong topology that it implies the corresponding convergence of a suitably renormalised version of our starting problem (1.2), which is the content of Theorem 2.10 below. Before we turn to this, we remark the following, which is the reason why we singled out the set $\mathfrak{S}_\circ$ above.

Lemma 2.5 *For $i \in \{\text{geo}, \text{Itô}\}$, the BPHZ characters $C^{\text{BPHZ}}_{\varepsilon,i}$ satisfy $C^{\text{BPHZ}}_{\varepsilon,i}(\tau) = 0$ for all $\tau \in \mathfrak{T}_- \setminus \mathfrak{S}_\circ$.*

Proof. For $\deg\tau \in \{-\frac{3}{2}^-, -\frac{1}{2}^-\}$, this follows from the fact that centred Gaussian random variables have vanishing odd moments. For the remaining symbols of degree $\deg\tau = 0^-$, but containing only two instances of $\circ$, this follows from the fact that in this case the corresponding processes $\hat{\mathbf{\Pi}}^{(\varepsilon)}_i\tau$ are odd in the sense that the identity

$$(\hat{\mathbf{\Pi}}^{(\varepsilon)}_i\tau)(t,-x) \stackrel{\text{law}}{=} -(\hat{\mathbf{\Pi}}^{(\varepsilon)}_i\tau)(t,x) \;,$$

holds in law as stochastic processes. □

Remark 2.6 The set of characters g of $\langle\!\langle\mathfrak{T}_-\rangle\!\rangle$ such that $g(\tau) = 0$ for $\tau \in \mathfrak{T}_- \setminus \mathfrak{S}_\circ$ forms a subgroup $\mathfrak{R}_\circ$ of the (reduced) renormalisation group $\mathfrak{R}$ studied in [BHZ18, Sec. 6.4.3]. Furthermore, this subgroup is canonically isomorphic to $(\mathcal{S}^*_\circ, +)$.

In particular, we see from the above list that the vertex-labels $\mathfrak{n}$ do not play any role as far as the BPHZ characters $C^{\text{BPHZ}}_{\varepsilon,i}$ are concerned since they vanish on all trees $T^{\mathfrak{n}}_{\mathfrak{f}}$ such that $\mathfrak{n} \not\equiv 0$. This suggests the introduction of the set $\mathfrak{T}_0 \subset \mathfrak{T}$ consisting of those trees $T^{\mathfrak{n}}_{\mathfrak{f}}$ with $\mathfrak{n} \equiv 0$, which we then simply denote by $T_{\mathfrak{f}}$. Finally, the following remark will be useful later on.

Lemma 2.7 *One also has $C^{\text{BPHZ}}_{\varepsilon,i}(\tau) = 0$ for $\tau \in \{$, , , , , $\}$.*

Proof. For $\tau \in \{$, $\}$, this follows from [HP15, Sec. 4.3], the argument for the remaining elements being essentially the same. The reason why this is the case is that, for each of these trees and for each way of partitioning the four instances of $\circ$ into two pairs that are then connected by two new edges, the resulting graph is connected but not two-connected.

For example, it follows from the general prescription [BHZ18] for the BPHZ renormalisation that one has the identity

$$\langle C^{\text{BPHZ}}_{\varepsilon,\text{geo}}, \cdot \rangle = \mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}} \cdot)(0) - \mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}} \cdot)(0)\,\mathbf{E}(\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}} \cdot)(0) = 0\;,$$

where the last identity follows from Wick's formula and we used $\circ$ and $\bullet$ to denote two instances of Ξ_i with different values for i. □

2.3 BPHZ theorem

In order to formulate the "BPHZ theorem" of [CH16, BCCH17] in our context, we first introduce a valuation $\Upsilon_{\Gamma,\sigma}$ which maps every linear combination of trees in $\mathfrak{T}_0$ into a smooth function $\mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ in the following way. For every tree $T_\mathfrak{f} \in \mathfrak{T}_0$, we set

$$\begin{aligned}(\Upsilon_{\Gamma,\sigma} T_\mathfrak{f})^\alpha(u,q) &= \\ &= \sum_{\beta:\mathbf{E}\to\{1,\ldots,d\}} \prod_{v\in V_T} \Bigg[\Big(\prod_{e\in\mathbb{E}^+_{\frown}(v)} \partial_{u^{\beta_e}}\Big)\Big(\prod_{e\in\mathbb{E}^+_{\frown}(v)} \partial_{q^{\beta_e}}\Big)(\bar\Upsilon^{\beta_{e_v}}_{\Gamma,\sigma}(v))(u,q)\Bigg]\end{aligned} \tag{2.6}$$

where

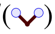
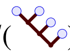
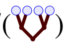
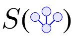
- $\mathbf{E}$ is the set of edges $e \in E_T$ of type $|$ or $|$, or in other words not of type $\circ_i$.
- For v a leaf, we set $\bar\Upsilon^\beta_{\Gamma,\sigma}(v)(u,q) = 1$, for v with an incoming edge of type $\circ_i$ for some i, we set $\bar\Upsilon^\beta_{\Gamma,\sigma}(v)(u,q) = \sigma^\beta_i(u)$ (the index i is uniquely determined since our rules R don't allow to have more than one incoming edge of type $\circ$ for the same vertex), while we set
$$\bar\Upsilon^\beta_{\Gamma,\sigma}(v)(u,q) = \Gamma^\beta_{\gamma\eta}(u)\, q^\gamma q^\eta\;, \tag{2.7}$$
otherwise.
- $\mathbb{E}^+_{\frown}(v)$ and $\mathbb{E}^+_{\frown}(v)$ are the sets of edges with decorations $|$ and $|$ respectively coming into $v \in V_T$.
- We use the convention $\beta_{e_v} = \alpha$ for $v = \varrho_T$, the root of T.

For instance, writing $\circ$ for Ξ_i and $\bullet$ for Ξ_j, one has
$$(\Upsilon_{\Gamma,\sigma} \cdot)^\alpha = 2\partial_\eta \Gamma^\alpha_{\beta\gamma}(u)\,\sigma^\eta_j(u)\,\sigma^\gamma_i(u)\,\partial_\zeta\sigma^\beta_i(u)\,\sigma^\zeta_j(u)\;.$$

The reason for the factor 2 appearing here is that (2.7) is differentiated (twice) with respect to the q-variable. By linearity, we extend the definition of $\Upsilon_{\Gamma,\sigma}$ to the linear vector space generated by $\mathfrak{T}_0$.

Remark 2.8 Note that for all $T_\mathfrak{f} \in \mathfrak{S}_\circ$, whenever a vertex isn't a leaf and has no incoming edge of type $\circ_i$, then it has exactly two incoming edges of type $|$. As a consequence, $\Upsilon_{\Gamma,\sigma} T_\mathfrak{f}$ only depends on u, in which case we drop the q-dependence from our notations.

We also assign to any $\tau \in \mathfrak{T}_0$ a symmetry factor $S(\tau)$ given by the number of tree automorphisms of τ (i.e. the number of graph isomorphisms of the corresponding tree T which furthermore preserve its root and all of its labels). For example, assuming that all instances of $\circ$ have the same index, we have

$S($  $) = 1\,,\quad S($  $) = 2\,,\quad S($  $) = 2\,,\quad S($  $) = 8\,,\quad S($  $) = 6\,.$

This endows $\mathcal{S}_\circ$ with a Hilbert space structure by postulating that, for $\tau, \sigma \in \mathfrak{T}_0$, one has $\langle \tau, \sigma \rangle = 0$ for $\tau \neq \sigma$ and $\langle \tau, \tau \rangle = S(\tau)$.

Remark 2.9 This allows us to identify $\mathcal{S}_\circ$ with its dual space $\mathcal{S}_\circ^*$ in the usual way. Combining this with Lemma 2.5 and Remark 2.6, we also identify $\mathfrak{R}_\circ$ with $(\mathcal{S}_\circ, +)$. This is in particular the reason why in Theorem 2.10 we can apply $\Upsilon_{\Gamma,\sigma}$ to the functional $C \in \mathfrak{R}_\circ$.

For some arbitrary but fixed $a \in (0, \frac{1}{2})$, we introduce the space $\mathscr{C}_\star^a$ of parabolic a-Hölder continuous functions $u\colon \mathbf{R}_+ \times S^1 \to \mathbf{R}^d$ with possible blow-up at finite time. Formally, we define $\mathscr{C}_\star^a$ as the set of pairs $(f, t_\star)$ where $t_\star \in (0, \infty]$ and $f\colon [0, t_\star) \times S^1 \to \mathbf{R}^d$ is a locally α-Hölder continuous function such that, if $t_\star < \infty$, its α-Hölder norm on $[0,t] \times S^1$ diverges as $t \uparrow t_\star$. This is a Polish space when endowed with the system of pseudo-metrics given by $\{d_L\}_{L \geq 1}$, where $d_L(F,G) = \|S_L(F) - S_L(G)\|_{\mathscr{C}^a([0,L])}$ with $S_L(f, t_\star)$ equal to the function f, stopped when either time or its space-time parabolic $\mathscr{C}^a$ norm exceeds L. (If $t_\star < \infty$, then this happens before time $t_\star$ by assumption, so this is a well-defined operation.) We also introduce the space $\tilde{\mathscr{C}}_\star^a$ of maps $\mathscr{C}^a(S^1, \mathbf{R}^d) \to \mathscr{C}_\star^a$ that are uniformly continuous on bounded sets, endowed with the topology of uniform convergence on bounded sets, as well as the space $\mathscr{B}_\star^a$ of continuous maps from $\mathscr{C}^a(S^1, \mathbf{R}^d)$ to the space of probability measures on $\tilde{\mathscr{C}}_\star^a$.

With this notation, the main results of [Hai14, CH16, BHZ18, BCCH17] can be combined into the following statement where $\varrho_\varepsilon * G$ should be read as a somewhat cumbersome way of simply writing G in the case $\varepsilon = 0$.

Theorem 2.10 *For every choice of smooth Γ, σ, K, h and every $a \in (0, \frac{1}{2} - \kappa)$, there exists a continuous "solution map" $\mathscr{A}(\Gamma, \sigma, K, h)\colon \mathscr{M}_\star \to \tilde{\mathscr{C}}_\star^a$ such that, for every $C \in \mathfrak{R}_\circ$, every continuous ζ, and setting*

$$\mathscr{M}_\star \ni Z = (\varepsilon, C \star \mathscr{L}_\varepsilon(\zeta))\,,$$

(with $\star$ as in (2.5)), $\mathscr{A}(\Gamma, \sigma, K, h)(Z)(u_0)$ solves the equation

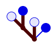
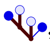
$$\begin{aligned}\partial_t u^\alpha = \partial_x^2 u^\alpha + \varrho_\varepsilon * [\Gamma_{\beta\gamma}^\alpha(u)\, \partial_x u^\beta \partial_x u^\gamma + K_\beta^\alpha(u)\, \partial_x u^\beta \\ + h^\alpha(u) + \sigma_i^\alpha(u)\, \zeta_i + (\Upsilon_{\Gamma,\sigma}^\alpha C)(u)]\,,\end{aligned} \tag{2.8}$$

for $t > 0$ and $u^\alpha(t, \cdot) = u_0^\alpha$ for $t \leq 0$. Furthermore, $\mathscr{A}(\Gamma, \sigma, K, h)(Z)$ is jointly continuous in Γ, σ, K, h and Z provided that the first four functions on $\mathbf{R}^d$ are equipped with the topology of convergence in the $\mathscr{C}^6$ topology on compact sets.

Proof. For $Z \in \mathscr{M}_0$, this is the content of [BCCH17, Thm. 2.20] (the symmetry factor $S(\tau)$ appearing in [BCCH17, Eq. 2.19] is created by the identification of $\mathcal{S}_\circ$ with $\mathcal{S}_\circ^*$ as already mentioned in Remark 2.9). The proof for general $Z \in \mathscr{M}_\star$ is virtually identical. The only difference is that the evolution now has some small amount of memory, so that one needs to keep track of it over a time interval of order ε^2 when restarting the fixed point argument. The continuity as $\varepsilon \to 0$ is taken care of by the time continuity built into the definition of the spaces $\mathscr{C}_\star^\alpha$. $\square$

Remark 2.11 The third argument K of $\mathscr{A}$ does not play much of a role from the point of view of the arguments in this article since it neither contributes to the counterterm $\Upsilon^\alpha_{\Gamma,\sigma} C$ nor is affected by it. We will therefore from now on only consider the case $K = 0$ and drop this argument in order to keep notations shorter.

2.4 Reduced trees

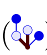
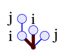
Define now $\mathfrak{S}_0$ in the same way as $\mathfrak{S}_\circ$, but with $m = 1$, so that there is only one "noise type", and denote by $S_0(\tau)$ the corresponding symmetry factor. We also define $\mathfrak{S}_0^{(k)}$ as the subset of trees in $\mathfrak{S}_0$ containing k noises where $k \in \{2,4\}$ and $\mathfrak{S}_0 = \mathfrak{S}_0^{(2)} \sqcup \mathfrak{S}_0^{(4)}$. Given $\tau \in \mathfrak{S}_0$, we write $\mathscr{P}_\tau^{(2)}$ for the set of all partitions of its noise set N_τ consisting of only two-elements blocks. (If τ happens to have an odd number of noises, then $\mathscr{P}_\tau^{(2)}$ is empty.) We then define $\hat{\mathfrak{S}}$ (resp. $\hat{\mathfrak{S}}_2$, $\hat{\mathfrak{S}}_4$) as the set of equivalence classes of pairs (τ, P) with $\tau \in \mathfrak{S}_0$ (resp. $\mathfrak{S}_0^{(2)}$, $\mathfrak{S}_0^{(4)}$) and $P \in \mathscr{P}_\tau^{(2)}$, where $(\tau, P) \sim (\tau', P')$ if there exists a tree isomorphism mapping τ to τ' and P to P'. For example, we want to make sure that we identify partitions of the form  and , where noises belong to the same block if and only if they have the same colour. We denote by $\mathcal{S}$ (resp. $\mathcal{S}_2$, $\mathcal{S}_4$) the vector space generated by those elements $(\tau, P) \in \hat{\mathfrak{S}}$ (resp. $\hat{\mathfrak{S}}_2$, $\hat{\mathfrak{S}}_4$) .

Given $\tau \in \mathfrak{S}_0$, we write $\mathscr{L}_\tau$ for the set of all maps $f\colon N_\tau \to \{\circ_1, \ldots, \circ_m\}$ and, for $f \in \mathscr{L}_\tau$, we write $f(\tau) \in \mathfrak{S}_\circ$ for the tree identical to τ, but with the noise type of each leaf $u \in N_\tau$ changed into $f(u)$. Given a partition P of N_τ and $f\colon N_\tau \to \{\circ_1, \ldots, \circ_m\}$, we also write $f \succ P$ if f is constant on each block of P.

The main result of this section is then that we can view our renormalisation constants as elements of $\mathcal{S}$. To see this, we first define $\iota : \mathcal{S} \to \mathcal{S}_\circ$

$$\iota(\tau, P) \stackrel{\text{def}}{=} \sum_{f\,:\, f \succ P} f(\tau)\;, \tag{2.9}$$

so that for example

$\iota($$) = \sum_{i,j}$  .

Lemma 2.12 *For every $\varepsilon > 0$ one has $\{C^{\text{BPHZ}}_{\varepsilon,\text{geo}}, C^{\text{BPHZ}}_{\varepsilon,\text{geo}}\} \subset \iota\mathcal{S}$.*

Proof. Fix $\varepsilon > 0$, $c \in \{C^{\text{BPHZ}}_{\varepsilon,\text{geo}}, C^{\text{BPHZ}}_{\varepsilon,\text{geo}}\}$. As a consequence of the definition of the scalar product on $\mathcal{S}_\circ$ given on page 21, we have the identity

$$c = \sum_{\tau \in \mathfrak{S}_\circ} \tau \frac{\langle c, \tau \rangle}{S(\tau)}\;, \tag{2.10}$$

where $S(\tau)$ denotes the symmetry factor defined in the previous section.

We then claim that one can rewrite (2.10) as

$$c = \sum_{\tau\in\mathfrak{S}_0}\sum_{f\in\mathscr{L}_\tau} f(\tau)\frac{\langle c, f(\tau)\rangle}{S_0(\tau)}\;. \tag{2.11}$$

Indeed, note that given $\tau_\circ \in \mathfrak{S}_\circ$ there exists a *unique* $\tau \in \mathfrak{S}_0$ for which there exists $f \in \mathscr{L}_\tau$ such that $f(\tau) = \tau_\circ$. The choice of f on the other hand is *not* unique since elements of $\mathfrak{S}_\circ$ are really equivalence classes modulo tree isomorphisms.

Writing $G(\tau)$ for the group of tree isomorphisms of τ and $G_f(\tau) \subset G(\tau)$ for those isomorphisms g such that $f \circ g = f$, (2.11) then follows from the fact that $S_0(\tau) = |G(\tau)|$ and $S(f(\tau)) = |G_f(\tau)|$, so that $S_0(\tau)/S(f(\tau))$ counts precisely the number of distinct maps $\bar f \in \mathscr{L}_\tau$ such that $\bar f(\tau) = f(\tau)$ (since $(f \circ g)(\tau) = f(\tau)$ for every $g \in G(\tau)$).

We further note that, given any $P \in \mathscr{P}_\tau$ and any $f \sim P$, i.e. such that the partition generated by f is *equal* to P, the exchangeability of the noises implies that $\langle c, f(\tau)\rangle$ is independent of the particular choice of such an f, so we denote it by $\langle c, (\tau, P)\rangle$ instead. With this notation, Wick's formula then implies the following result, the proof of which is given below.

Lemma 2.13 *For every $\tau \in \mathfrak{S}_0$, every $\varepsilon > 0$, every $c \in \{C^{\textsc{bphz}}_{\varepsilon,\mathrm{geo}}, C^{\textsc{bphz}}_{\varepsilon,\mathrm{geo}}\}$, and every $f \in \mathscr{L}_\tau$, one has the identity*

$$\langle c, f(\tau)\rangle = \sum_{P\in\mathscr{P}^{(2)}_\tau : f\succ P} \langle c, (\tau, P)\rangle\;.$$

As a consequence of this lemma, we conclude that

$$c = \sum_{\tau\in\mathfrak{S}_0}\sum_{P\in\mathscr{P}^{(2)}_\tau}\frac{\langle c, (\tau,P)\rangle}{S_0(\tau)}\sum_{f\,:\,f\succ P} f(\tau) = \sum_{\tau\in\mathfrak{S}_0}\sum_{P\in\mathscr{P}^{(2)}_\tau}\frac{\langle c, (\tau,P)\rangle}{S_0(\tau)}\iota(\tau,P)\;, \tag{2.12}$$

as claimed. □

Proof of Lemma 2.13. Consider the case $i = \mathrm{geo}$ first and fix τ and f as in the statement, as well as $\varepsilon > 0$. Given a subset K of N_τ, we denote by E^K_f the function on $(\mathbf{R}^2)^K$ given by

$$E^K_f(z) = \mathbf{E}\Big(\prod_{u\in K}\xi^{(\varepsilon)}_{f(u)}(z_u)\Big)\;. \tag{2.13}$$

Similarly, for any $P \in \mathscr{P}^{(2)}(K)$, the set of partitions of K into sets of size two, we write $E^K_P = E^K_g$ for any g which generates the partition P. Wick's formula can then be stated as

$$E^K_f = \sum_{P\in\mathscr{P}^{(2)}(K)\,:\,f\succ P} E^K_P\;.$$

(Note that this sum is empty if $|K|$ is odd.)

It also follows from Remark 2.3 that there exist finitely many distributions η_j on $(\mathbf{R}^2)^{N_\tau}$ and partitions O_j of N_τ such that

$$\langle c, f(\tau)\rangle = \sum_j \int \Big(\prod_{K\in O_j} E_f^K(z|K)\Big)\, \eta_i(dz)\;,$$

where, for $z \in (\mathbf{R}^2)^{N_\tau}$, we denote by $z|K$ its restriction to K, so that

$$\langle c, f(\tau)\rangle = \sum_j \sum_{\substack{P\in\mathscr{P}_\tau^{(2)}\\ P\prec O_j \,\&\, P\prec f}} \int \Big(\prod_{K\in O_j} E_P^K(z|K)\Big)\, \eta_i(dz)\;,$$

where $P \prec O_j$ means that P refines O_j. It now suffices to note that if, given $P \in \mathscr{P}_\tau^{(2)}$, we write f_P for any one function with $f_P \sim P$, then we have

$$\langle c, f(\tau)\rangle = \sum_{P\in\mathscr{P}_\tau^{(2)}\,:\,P\prec f} \sum_j \int \Big(\prod_{K\in O_j} E_{f_P}^K(z|K)\Big)\, \eta_i(dz) = \sum_{P\in\mathscr{P}_\tau^{(2)}} \langle c, (\tau, P)\rangle\;,$$

as desired. This is because, if there happens to be some j for which P does not refine O_j, then one of the factors $E_{f_P}^K(z|K)$ for $K \in O_j$ necessarily vanishes. The proof for $i =$ Itô is identical, the only difference being that E_f^K is now a distribution (a sum of products of Dirac distributions since the $\xi^{(\varepsilon)}$ in (2.13) should be replaced by ξ's), while the η_i are smooth ε-dependent functions. □

Remark 2.14 Given a tree $\tau \in \mathfrak{S}_0$ and a pairing P of its noises, we have a natural symmetry factor $S(\tau, P)$ which counts the number of tree isomorphisms of τ that keep the pairing P intact. With this notation, we can then rewrite (2.12) as

$$c = \sum_{(\tau,P)} \iota(\tau, P)\frac{\langle c, (\tau, P)\rangle}{S(\tau, P)}\;. \tag{2.14}$$

Indeed, if we denote by $N(\tau, P)$ the number of distinct pairings Q of the noises of τ such that $(\tau, Q) = (\tau, P)$ modulo tree isomorphism, then one has $N(\tau, P)/S(\tau) = 1/S(\tau, P)$. This is of course equivalent to $N(\tau, P) = S(\tau)/S(\tau, P)$, which follows from the fact that $S(\tau, P)$ is the stabiliser of $S(\tau)$ with respect to P if we view the group of tree isomorphisms as acting on the set of all pairings of the noises of τ while $N(\tau, P)$ is precisely the size of the orbit of P under that action.

Remark 2.15 The space $\mathcal{S}$ is naturally endowed with the scalar product such that elements of $\hat{\mathfrak{S}}$ are orthogonal and such that $\|(\tau, P)\|^2 = S(\tau, P)$. This definition of the scalar product on $\mathcal{S}$ is consistent with (2.14). We also extend the valuation $\Upsilon_{\Gamma,\sigma}$ defined in Section 2.3 to $\mathcal{S}$ by composition with ι.

3 Symmetry properties of the solution map

The goal of this section is to show that it is possible to construct solution maps having the various symmetries laid out in Theorem 1.2. We do this by analysing two different approximation schemes corresponding to the models $\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}}$ and $\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}$ constructed above. We therefore define

1. $U^{\text{geo}}_\varepsilon(\Gamma,\sigma,h) \in \mathscr{B}^a_\star$ as the law of $\mathscr{A}(\Gamma,\sigma,h)(\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}})(u_0)$
2. $U^{\text{Itô}}_\varepsilon(\Gamma,\sigma,h) \in \mathscr{B}^a_\star$ as the law of $\mathscr{A}(\Gamma,\sigma,h)(\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}})(u_0)$
3. $U^{\text{BPHZ}}(\Gamma,\sigma,h) \in \mathscr{B}^a_\star$ as the law of $\mathscr{A}(\Gamma,\sigma,h)(\mathbf{\Pi}^{\text{BPHZ}})(u_0)$

where $\mathscr{B}^a_\star$ is defined on page 21, $\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}}$ and $\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}$ are defined in (2.3), and $\mathbf{\Pi}^{\text{BPHZ}}$ is the limit of both $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{geo}}$ and $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{Itô}}$ given by Theorem 2.4, where $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{geo}}$ and $\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{Itô}}$ are defined by (2.4).

3.1 Symmetry properties of the approximations

The reason for considering these two different approximations is that each of them satisfies one of the two symmetries, as formulated in the following result.

Proposition 3.1 *For every diffeomorphism φ of $\mathbf{R}^d$, the identity*

$$\varphi \bullet U^{\text{geo}}_\varepsilon(\Gamma,\sigma,h) = U^{\text{geo}}_\varepsilon(\varphi\bullet\Gamma,\varphi\bullet\sigma,\varphi\bullet h)\,, \tag{3.1}$$

holds for all smooth choices of Γ, σ and h. Furthermore, for all smooth $\bar\sigma$ such that

$$\bar\sigma^\alpha_i\bar\sigma^\beta_i = \sigma^\alpha_i\sigma^\beta_i \tag{3.2}$$

holds, one has $U^{\text{Itô}}_\varepsilon(\Gamma,\sigma,h) = U^{\text{Itô}}_\varepsilon(\Gamma,\bar\sigma,h)$.

Proof. The definition of $U^{\text{geo}}_\varepsilon$ implies that $U^{\text{geo}}_\varepsilon(\Gamma,\sigma,h)(u_0)$ is the law of the random PDE (1.5). The identity (3.1) then simply states that the solutions to this random PDE behave as expected under changes of variables, which follows immediately from the usual rules of calculus which can be applied pathwise since all objects under consideration are smooth. We therefore only need to consider the claim for $U^{\text{Itô}}_\varepsilon$.

Write $\mathbb{Q}$ for the set of pairs $(\sigma,\bar\sigma)$ of $d\times m$ real-valued matrices such that $\sigma\sigma^\top = \bar\sigma\bar\sigma^\top$. It is a simple consequence of the reduced LQ factorisation that there exists a map $\Sigma\colon \mathbb{Q}\to SO(m)$ such that, for every $(\sigma,\bar\sigma)\in\mathbb{Q}$, one has $\sigma\Sigma(\sigma,\bar\sigma) = \bar\sigma$. By the Kuratowski–Ryll–Nardzewski measurable selection theorem (see for example [Bog07, Vol. II, p. 36]), we can furthermore choose Σ to be Borel measurable.

For u_0 fixed, let now u be the maximal solution to (2.8) as in the proof of Theorem 2.10. Write $\mathscr{H} = L^2(S^1,\mathbf{R}^m)$, and let W by the $\mathscr{H}$-cylindrical Wiener process such that, for every smooth test function $\varphi\colon [-1,\infty)\times S^1\to\mathbf{R}^m$ (canonically identified with a function $[-1,\infty)\to\mathscr{H}$), one has

$$\int_{\mathscr{R}} \langle\varphi(t,\cdot), dW(t)\rangle = \xi_i(\varphi_i)\;.$$

Since we assume that ϱ is non-anticipative, the solution u is smooth and adapted, so that the distributional products $\sigma_i^\alpha(u)\,\xi_i$ coincide with their interpretations as Itô integrals with respect to W as above.

Fix $L > 0$ and let τ be the stopping time given by the first time at which the space-time α-Hölder norm of u exceeds L. For $\Sigma\colon \mathbb{Q} \to SO(m)$ as before, we then set

$$Q(t)(x) = \begin{cases} \Sigma(\sigma(u(t,x)),\sigma(u(t,x)))^\top & \text{if } t \le \tau, \\ \text{id} & \text{otherwise,} \end{cases}$$

so that $t \mapsto Q(t)$ is a progressively measurable process with values in $\mathbb{O}_m \stackrel{\text{def}}{=} \mathscr{B}_b(S^1, SO(m))$. (For $t \le 0$, we use the convention $u(t) = u_0$.) Since pointwise multiplication by an element of $\mathbb{O}_m$ yields a unitary operator on $\mathscr{H}$, the process

$$\bar W(t) \stackrel{\text{def}}{=} \int_{-1}^t Q(s)\,dW(s)$$

is again a cylindrical Wiener process on $\mathscr{H}$, see [DPZ14], so that in particular the laws of W and of $\bar W$ coincide.

Denote by $\bar\xi$ the corresponding distribution and note that one then has the almost sure distributional identity

$$\bar\sigma_i^\alpha(u)\,\bar\xi = \sigma_i^\alpha(u)\,\xi_i\;,$$

as a straightforward consequence of the fact that, for any bounded $\mathscr{H}$-valued progressively measurable function A one has the identity $\int_0^t \langle A(s), d\bar W(s)\rangle = \int_0^t \langle Q^*(s)A(s), dW(s)\rangle$. The claim now follows at once since we are working with the model $\mathbf{\Pi}^{(\varepsilon)}{}_{\text{Itô}}$ given by the canonical lift $\mathscr{L}_\varepsilon(\xi)$, so that the reconstruction applied to any product is simply given by the product of the reconstructions. □

It is then natural to define the following two subspaces of $\mathcal{S}$ which correspond to the two invariance / equivariance properties appearing in this statement. (Recall that $\Upsilon_{\Gamma,\sigma}$ is defined on $\mathcal{S}$ by Remark 2.15.)

Definition 3.2 The space $\mathcal{S}_{\text{geo}} \subset \mathcal{S}$ consists of those elements τ such that, for all $d, m \ge 1$ and all choices of Γ and σ as above and all diffeomorphisms φ of $\mathbf{R}^d$ that are homotopic to the identity, one has the identity

$$\varphi \bullet (\Upsilon_{\Gamma,\sigma}\tau) = \Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma}\tau\;. \tag{3.3}$$

Similarly, the space $\mathcal{S}_{\text{Itô}} \subset \mathcal{S}$ consists of those elements τ such that, for all d, m, Γ, σ and $\bar\sigma$ as above with $(\sigma,\bar\sigma) \in \mathbb{Q}$, the identity $\Upsilon_{\Gamma,\sigma}\tau = \Upsilon_{\Gamma,\bar\sigma}\tau$ holds.

Remark 3.3 We do for the moment restrict ourselves to equivariance under diffeomorphisms homotopic to the identity. We will however see in Remark 6.16 below that, for $\tau \in \mathcal{S}_{\text{geo}}$, the identity (3.3) automatically holds for all diffeomorphisms.

With these notations at hand, we have the following immediate corollary of Proposition 3.1.

Corollary 3.4 *Assume that, for* $i \in \{\text{geo}, \text{Itô}\}$*, there exist sequences* $\tau_i^{(\varepsilon)} \in \mathcal{S}_i$ *such that* $\tau_i^{(\varepsilon)} - C_{\varepsilon,i}^{\text{BPHZ}}$ *converges to a finite limit* α_i *as* $\varepsilon \to 0$*. Then the limits*

$$U^i \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \hat{U}_\varepsilon^i\;, \quad \hat{U}_\varepsilon^i(\Gamma,\sigma,h) = U_\varepsilon^i(\Gamma,\sigma,h+\Upsilon_{\Gamma,\sigma}\tau_i^{(\varepsilon)})\;,$$

exist, satisfy their respective equivariance properties as in Proposition 3.1 (except that the equivariance under the diffeomorphisms group is a priori restricted to those homotopic to the identity), and are equal to

$$U^i(\Gamma,\sigma,h) = U^{\text{BPHZ}}(\Gamma,\sigma,h+\Upsilon_{\Gamma,\sigma}\alpha_i). \tag{3.4}$$

Proof. The fact that the limits exist follows from the continuity with respect to (h, Z) stated in Theorem 2.10, combined with our assumption and Theorem 2.4. The fact that U^{geo} is equivariant under the diffeomorphism group follows immediately from the fact that this is true for $\hat{U}_\varepsilon^{\text{geo}}$ for every ε, combined with the fact that the diffeomorphism group acts continuously on $\mathcal{B}_\star^a$. The argument for $U^{\text{Itô}}$ is similar.

Finally, it follows from Theorem 2.10 and the definition of U^{BPHZ} that

$$\begin{aligned} U^{\text{BPHZ}}(\Gamma,\sigma,h) &= \lim_{\varepsilon\to 0} U_\varepsilon^i\big(\Gamma,\sigma,h+\Upsilon_{\Gamma,\sigma}C_{\varepsilon,i}^{\text{BPHZ}}\big) \\ &= \lim_{\varepsilon\to 0} \hat{U}_\varepsilon^i\Big(\Gamma,\sigma,h+\Upsilon_{\Gamma,\sigma}(C_{\varepsilon,i}^{\text{BPHZ}} - \tau_i^{(\varepsilon)})\Big) = U^i(\Gamma,\sigma,h-\Upsilon_{\Gamma,\sigma}\alpha_i) \end{aligned}$$

and (3.4) is proved. □

Therefore, in order to construct a single solution map satisfying both relevant symmetry properties, we need to show that we can find such sequences in a way such that $\tau_{\text{geo}}^{(\varepsilon)} - C_{\varepsilon,\text{geo}}^{\text{BPHZ}}$ and $\tau_{\text{Itô}}^{(\varepsilon)} - C_{\varepsilon,\text{Itô}}^{\text{BPHZ}}$ both converge to the *same* limit. Given the additional constraint that these sequences need to lie in the (potentially quite small) subspaces $\mathcal{S}_{\text{geo}}$ and $\mathcal{S}_{\text{Itô}}$, it is not clear at all that this is possible. (It is not even clear a priori that one can find sequences as in the statement of the corollary!) One ingredient of our argument is the fact that the limiting solution map U^{BPHZ} is injective in its last argument, which is the content of the next section.

3.2 Injectivity of the law

In this section, we show that for any fixed (Γ,σ), the map $h \mapsto U^{\text{BPHZ}}(\Gamma,\sigma,h)$ is injective. Note that this is a strictly stronger notion of injectivity than the one for the map $\mathcal{A}(\Gamma,\sigma,h)$ obtained in Theorem 2.10 that maps the pair $(u_0, \mathbf{\Pi})$ to the corresponding solution. Indeed, as already announced in the introduction, the map $(\sigma,h) \mapsto U^{\text{BPHZ}}(\Gamma,\sigma,h)$ is *not* injective since we get the same law for any two choices of σ that define the same g provided that h is adjusted accordingly, while we expect $(\sigma,h) \mapsto \mathcal{A}(\Gamma,\sigma,h)$ to be injective.

Theorem 3.5 *For all dimensions d and m and smooth choices of Γ and σ, the map*

$$h \mapsto U^{\text{BPHZ}}(\Gamma, \sigma, h)$$

is injective as a map from $\mathcal{C}^6$ to $\mathcal{B}^\alpha_\star$.

Our main ingredient in the proof of this result is the following.

Lemma 3.6 *Let $\boldsymbol{\Pi}^{\text{BPHZ}}$ denote the BPHZ model as in Theorem 2.4, let $u_0 \in \mathcal{C}^\alpha(S^1)$, and let V be the solution in $\mathcal{D}^\gamma$ to the fixed point problem*

$$V^\alpha = Pu_0^\alpha + \mathcal{P}\mathbf{1}_+(\Gamma^\alpha_{\beta,\gamma}(V)\, \mathscr{D}V^\beta\, \mathscr{D}V^\gamma + h^\alpha(V) + \sigma_i^\alpha(V)\, \Xi_i)\ . \tag{3.5}$$

Write $u^\alpha = \mathcal{R}V^\alpha$, which is nothing but the continuous function given by the $\mathbf{1}$-component of V^α. Fix furthermore some $r > 0$ and set

$$\tau_r = \inf\{t > 0\ :\ \|u - Pu_0\|_{\mathcal{C}^\alpha_t} \geq r\}\ , \tag{3.6}$$

where $\|\cdot\|_{\mathcal{C}^\alpha_t}$ denotes the space-time Hölder norm on $[0,t] \times S^1$. Then, the random $\mathbf{R}^d$-valued distribution η_r on $\mathbf{R} \times S^1$ given by

$$\eta_r = \mathcal{R}\mathbf{1}_{[0,\tau_r]}(\sigma_i^\alpha(V)\, \Xi_i) \tag{3.7}$$

is such that, for every (deterministic) test function ψ, $\eta_r(\psi)$ coincides with the Itô integral

$$\int_0^{\tau_r} \langle \sigma_i^\alpha(v_s)\psi, dW_i(s)\rangle\ , \tag{3.8}$$

with $(t,x) \mapsto v_t(x)$ given by $\mathcal{R}V$ (which is guaranteed to be a continuous function and can therefore be evaluated at fixed time slices) and W_i the L^2-cylindrical Wiener process associated to the noise $\xi_i = \mathcal{R}\Xi_i$ as in the proof of Proposition 3.1. In particular, these random variables have vanishing expectation.

Proof. Note first that the quantity (3.8) is well-defined since $s \mapsto v_s$ is continuous and adapted.

To show that this is the case, we use the fact that $\eta_r(\psi) = \lim_{\varepsilon\to 0} \eta_r^\varepsilon(\psi)$, where $\eta_r^{(\varepsilon)}$ is defined as above, but with both the reconstruction operator $\mathcal{R}$, the modelled distribution V, and the stopping time τ_r obtained from the model $\hat{\boldsymbol{\Pi}}^{(\varepsilon)}_{\text{Itô}}$. It then follows from [BCCH17] that for any fixed $\varepsilon > 0$ one has the identity

$$\eta_r^{(\varepsilon)}(\psi) = \int_0^{\tau_r} \langle \sigma_i^\alpha(v_s)\psi, dW_i(s)\rangle + \int_0^{\tau_r} \langle \psi, \Upsilon^\alpha_{\Gamma,\sigma} P_\circ C^{\text{BPHZ},\varepsilon}_{\text{Itô}}\rangle\, ds\ ,$$

where $P_\circ\colon \mathcal{S} \to \mathcal{S}$ is the projection onto the subspace spanned by those symbols that have a noise edge incident to their root.

On the other hand, we already noted in the proof of Theorem 2.4 that $P_\circ C^{\text{BPHZ},\varepsilon}_{\text{Itô}} = 0$, whence the claim follows by continuity and the stability of the Itô integral. □

Before we turn to the proof of Theorem 3.5, we introduce a larger regularity structure than the one considered so far. The difference is that we allow for two different edge types, $\textcolor{blue}{|}$ and $\textcolor{cyan}{|}$, and we will consider models that are admissible in the sense that edges of type $\textcolor{blue}{|}$ represent the kernel K and edges of type $\textcolor{cyan}{|}$ represent K_ε. As in Section 2.1, we identify $\textcolor{blue}{|}$ with $(\textcolor{blue}{|}, 0)$ and we write $\textcolor{red}{\vdots} = (\textcolor{cyan}{|}, (0,1))$. We then extend our rule R to this larger type set by setting

$$\begin{aligned}\hat R(\textcolor{blue}{|}) &= \{(\textcolor{blue}{|}^k, \circ_i), (\textcolor{red}{\mathbf{|}}^\ell, \textcolor{blue}{|}^k) \,:\, k \geq 0,\ \ell \in \{0,1,2\},\ i \in \{1,\ldots,m\}\}\;,\\ \hat R(\textcolor{cyan}{|}) &= \{(\textcolor{blue}{|}^k, \textcolor{cyan}{|}^{\bar k}, \circ_i), (\textcolor{red}{\mathbf{|}}^\ell, \textcolor{red}{\vdots}^{\bar\ell}, \textcolor{blue}{|}^k, \textcolor{cyan}{|}^{\bar k}) \,:\, k, \bar k \geq 0,\ \ell + \bar\ell \in \{0,1,2\},\ i \in \{1,\ldots,m\}\}\;.\end{aligned}$$

We write $\hat{\mathfrak{T}}$ for the corresponding set of labelled trees and $\hat{\mathcal{T}}$ for the vector space generated by $\hat{\mathfrak{T}}$. This is turned into a regularity structure $\hat{\mathscr{T}}$ as in [BHZ18, Sec. 6.4] (the "reduced regularity structure" in that lingo) and, similarly to above, we write $\hat{\mathscr{M}}$ for the space of all models for $\hat{\mathscr{T}}$ and $\hat{\mathscr{M}}_\varepsilon$ for the corresponding space of *admissible* models with $\textcolor{blue}{|}$ assigned to K and $\textcolor{cyan}{|}$ assigned to K_ε. Since $\hat R$ extends R, it follows that we have a canonical inclusion $\mathcal{T} \subset \hat{\mathcal{T}}$, as well as a canonical projection $\pi_0 \colon \hat{\mathscr{M}}_\varepsilon \to \mathscr{M}_0$ obtained by restricting a given model to $\mathcal{T}$. One also has a canonical projection $\hat\pi \colon \hat{\mathcal{T}} \to \mathcal{T}$ obtained by changing all edges of type $\textcolor{cyan}{|}$ / $\textcolor{red}{\vdots}$ into edges of type $\textcolor{blue}{|}$ / $\textcolor{red}{\mathbf{|}}$. It is immediate from the definition of $\hat R$ that if τ is a labelled tree conforming to the rule $\hat R$, then $\hat\pi\tau$ conforms to the rule R, so that $\hat\pi$ does indeed map $\hat{\mathcal{T}}$ to $\mathcal{T}$. Since furthermore both $\textcolor{blue}{|}$ and $\textcolor{cyan}{|}$ are assigned to K in the space $\hat{\mathscr{M}}_0$, this yields a canonical inclusion $\iota \colon \mathscr{M}_0 \hookrightarrow \hat{\mathscr{M}}_0$ obtained by right composing with $\hat\pi$. The inclusion ι is a right inverse for the projection π_0.

For $\varepsilon > 0$, we also have a natural extension map $\mathscr{E}_\varepsilon \colon \mathscr{M}_0 \to \hat{\mathscr{M}}_\varepsilon$, also forming a right inverse for π_0, which is defined as follows. Given a model $\mathbf{\Pi} \in \mathscr{M}_0$, we define $\hat{\mathbf{\Pi}} = \mathscr{E}_\varepsilon(\mathbf{\Pi})$ in such a way that $\hat{\mathbf{\Pi}}$ is admissible and such that, whenever $\tau \in \hat{\mathcal{T}}$ is of the form $\tau = \bar\tau \cdot \tau_1 \cdots \tau_k$ for $\bar\tau \in \mathcal{T}$ and $\tau_i \in \hat{\mathcal{T}}$ such that their root is only incident to exactly one edge of type $\textcolor{cyan}{|}$ / $\textcolor{red}{\vdots}$ (the cases $k = 0$ and / or $\bar\tau = \mathbf{1}$ are allowed), then

$$\hat{\mathbf{\Pi}}\tau = \mathbf{\Pi}\bar\tau \cdot \hat{\mathbf{\Pi}}\tau_1 \cdots \hat{\mathbf{\Pi}}\tau_k\;.$$

Since K_ε is smooth, it follows that $\hat{\mathbf{\Pi}}\tau_i$ is smooth for any such symbol, so this product is simply a product between a distribution and a number of smooth functions. Furthermore, this covers all of $\hat{\mathcal{T}}$, so that it determines $\hat{\mathbf{\Pi}}$ uniquely.

Since one can easily verify that $\hat R$ is complete (as a consequence of the fact that R is complete), we also have a renormalisation group $\hat{\mathfrak{R}}$ associated to $\hat{\mathscr{T}}$, with elements of $\hat{\mathfrak{R}}$ identified with characters of the free algebra $\langle\!\langle \hat{\mathfrak{T}}_- \rangle\!\rangle$ generated by the trees $\tau \in \hat{\mathfrak{T}}_-$ of strictly negative degree, as well as a subgroup $\hat{\mathfrak{R}}_\circ$ defined analogously to $\mathfrak{R}_\circ$. The canonical inclusion $\mathfrak{T}_- \subset \hat{\mathfrak{T}}_-$ allows to identify $\mathfrak{R}$ with a subgroup of $\hat{\mathfrak{R}}$. We also have a natural normal subgroup $\mathfrak{R}^\perp \subset \hat{\mathfrak{R}}$ consisting of those group elements which leave $\mathcal{T}$ invariant. This corresponds precisely to the linear functionals that vanish on $\mathfrak{T}_-$. Similarly, we define the subgroup $\mathfrak{R}_\circ^\perp = \hat{\mathfrak{R}}_\circ \cap \mathfrak{R}^\perp$.

Note that the extension operator $\mathscr{E}_\varepsilon$ is only well-defined for $\varepsilon > 0$ and does *not* converge to the operator ι as $\varepsilon = 0$. However, we have the following result which is the other main ingredient in the proof of Theorem 3.5.

Lemma 3.7 *Let $\boldsymbol{\Pi}^{\text{BPHZ}}$ be the (random) BPHZ model on $\mathscr{T}$ constructed in Theorem 2.4. Then, there exists a sequence of elements $C_\varepsilon \in \mathfrak{R}_\circ^\perp$, so that the sequence of models*

$$\hat{\boldsymbol{\Pi}}_\varepsilon = C_\varepsilon \star \mathscr{E}_\varepsilon(\boldsymbol{\Pi}^{\text{BPHZ}}) \;,$$

converges in $\hat{\mathscr{M}}$ to the model $\iota\boldsymbol{\Pi}^{\text{BPHZ}}$.

Proof. We consider the models $\boldsymbol{\Pi}_{\varepsilon,\delta} = \mathscr{E}_\varepsilon(\hat{\boldsymbol{\Pi}}^{(\delta)}_{\text{geo}})$ and look at the sequence

$$\hat{\boldsymbol{\Pi}}_{\varepsilon,\delta} = C^{\text{BPHZ}}_{\varepsilon,\delta} \star \boldsymbol{\Pi}_{\varepsilon,\delta} \;,$$

where $C^{\text{BPHZ}}_{\varepsilon,\delta}$ is the corresponding BPHZ character which is characterised by the fact that $\hat{\boldsymbol{\Pi}}_{\varepsilon,\delta}$ vanishes on elements of negative degree.

As in Remark 2.3, we have the explicit formula

$$C^{\text{BPHZ}}_{\varepsilon,\delta} = \hat{\mathscr{A}}^t g_{\varepsilon,\delta} \;,$$

where $\hat{\mathscr{A}}^t \colon \hat{\mathfrak{T}}_- \to \langle\!\langle \hat{\mathfrak{T}} \rangle\!\rangle$ is the twisted antipode for the regularity structure $\hat{\mathscr{T}}$ and $g_{\varepsilon,\delta}$ is given by

$$g_{\varepsilon,\delta}(\tau) = \mathbf{E}(\boldsymbol{\Pi}_{\varepsilon,\delta}\tau)(0) \;.$$

It follows immediately from the definition of the twisted antipode [BHZ18, Eq. 6.24] that $\hat{\mathscr{A}}^t$ coincides with $\mathscr{A}^t$ on $\mathfrak{T}_- \subset \hat{\mathfrak{T}}_-$. Since we furthermore have $g_{\varepsilon,\delta}(\tau) = 0$ for $\tau \in \mathfrak{T}_-$ (this is precisely what characterises $\hat{\boldsymbol{\Pi}}^{(\delta)}_{\text{geo}}$, which furthermore coincides with $\boldsymbol{\Pi}_{\varepsilon,\delta}$ on those elements), it follows that $C^{\text{BPHZ}}_{\varepsilon,\delta} \in \mathfrak{R}_\circ^\perp$.

By stability of the BPHZ model, $\hat{\boldsymbol{\Pi}}_{\varepsilon,\delta}$ converges to $\iota\boldsymbol{\Pi}^{\text{BPHZ}}$ as $\varepsilon, \delta \to 0$ in whichever way one takes these limits, so it only remains to show that $g_{\varepsilon,\delta}$ (and therefore also $C^{\text{BPHZ}}_{\varepsilon,\delta}$) has a limit as $\delta \to 0$ for any fixed ε. For this, we note that the only elements $\tau \in \hat{\mathfrak{T}}_-$ for which $\boldsymbol{\Pi}_{\varepsilon,\delta}\tau$ is not a stationary process are $\otimes_i$, as well as those modelled (in the sense that some lines may be replaced by dotted lines) on the first four elements of the last line of the table of symbols on page 16. These however are odd (in law) in the spatial variable, so that

$$g_{\varepsilon,\delta}(\tau) = 0 = \mathbf{E}(\boldsymbol{\Pi}_{\varepsilon,\delta}\tau)(\varphi) \;,$$

for any test function φ that is even in the spatial variable. It follows that, for any such test function integrating to 1, one has

$$g_{\varepsilon,\delta}(\tau) = \mathbf{E}(\boldsymbol{\Pi}_{\varepsilon,\delta}\tau)(\varphi) \;.$$

Since the model $\boldsymbol{\Pi}_{\varepsilon,\delta}$ converges to the finite limit $\mathscr{E}_\varepsilon(\boldsymbol{\Pi}^{\text{BPHZ}})$ as $\delta \to 0$, the claim follows. □

Remark 3.8 Since $C_\varepsilon \in \mathfrak{R}_\circ^\perp$ and since $(\pi_0 \circ \mathscr{E}_\varepsilon)(\boldsymbol{\Pi}^{\text{BPHZ}}) = \boldsymbol{\Pi}^{\text{BPHZ}}$, one has $\pi_0(\hat{\boldsymbol{\Pi}}_\varepsilon) = \boldsymbol{\Pi}^{\text{BPHZ}}$ for every $\varepsilon > 0$.

We now have all the ingredients in place to give a proof of Theorem 3.5.

Proof of Theorem 3.5. Our main ingredient is the fact that we can find a measurable function $F_{\Gamma,h}\colon \mathscr{C}^\alpha_\star \to \mathscr{D}'$ with the property that, if we denote by $\bar u$ a random variable with law $U^{\text{BPHZ}}(\Gamma,\sigma,\bar h)(u_0)$, then the law of $F_{\Gamma,h}(\bar u)$ coincides with that of the random variable

$$\eta_r + \mathbf{1}_{[0,\tau_r]}(h(\bar u) - \bar h(\bar u)) \;. \tag{3.9}$$

Assume for the moment that this is indeed the case and, for given Γ and σ, choose any two vector fields $h \neq \bar h$. In particular there exist $v, w \in \mathbf{R}^d$ and $r > 0$ such that

$$\langle w, h(v') - \bar h(v')\rangle \geq 1 \;, \tag{3.10}$$

for all v' such that $|v' - v| \leq r$. Write u for a random variable with law $U^{\text{BPHZ}}(\Gamma,\sigma,h)(v)$, $\bar u$ for a random variable with law $U^{\text{BPHZ}}(\Gamma,\sigma,\bar h)(v)$, and let $\tau_r(u)$ be defined as in (3.6) with $u_0 = v$.

Since $\tau_r > 0$ almost surely, we can find δ sufficiently small so that $\mathbf{P}(\tau_r(u) > \delta) > \frac12$ and $\mathbf{P}(\tau_r(\bar u) > \delta) > \frac12$. Choose then a smooth real-valued test function φ integrating to 1 and supported on $[0,\delta]\times S^1$. By (3.9), (3.10), the fact that η_r has vanishing expectation, and our choice of δ, it then follows that

$$\langle F_{\Gamma,h}(u), w\varphi\rangle = 0 \;, \qquad \langle F_{\Gamma,h}(\bar u), w\varphi\rangle > \frac12 \;,$$

so that the laws of u and $\bar u$ are indeed distinct as claimed.

It remains to show that one can construct an $F_{\Gamma,h}$ such that (3.9) holds. For this, consider the (unique) solutions to the following system of equations

$$\begin{aligned} V^\alpha &= Pu_0^\alpha + \mathscr{P}\mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta,\gamma}(V)\,\mathscr{D}V^\beta\,\mathscr{D}V^\gamma + h^\alpha(V) + \sigma^\alpha_i(V)\,\Xi_i) \;, \\ \bar V^\alpha_\varepsilon &= \varrho_\varepsilon * Pu_0^\alpha + \mathscr{P}_\varepsilon\mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta,\gamma}(V)\,\mathscr{D}V^\beta\,\mathscr{D}V^\gamma + h^\alpha(V) + \sigma^\alpha_i(V)\,\Xi_i) \;, \end{aligned} \tag{3.11}$$

where τ_r is as above, based on $u = \mathscr{R}V$. It follows from the definitions that, for any random model $\hat{\mathbf{\Pi}} \in \hat{\mathscr{M}}_\varepsilon$ such that $\pi_0(\hat{\mathbf{\Pi}}) = \mathbf{\Pi}^{\text{BPHZ}}$ (in law), the law of $\mathscr{R}V^\alpha$ coincides with the law of $U^{\text{BPHZ}}(\Gamma,\sigma,\bar h)(u_0)$ (and V is indeed independent of ε). Furthermore, the identity $\mathscr{R}\bar V^\alpha_\varepsilon = \varrho_\varepsilon * \mathscr{R}V^\alpha$ holds for every model $\hat{\mathbf{\Pi}} \in \hat{\mathscr{M}}_\varepsilon$. Finally, for any model of the for $\iota\mathbf{\Pi}$ for some $\mathbf{\Pi} \in \mathscr{M}_0$, $\mathscr{R}\bar V_0 = \mathscr{R}V$. In fact, a stronger statement is true, namely

$$\hat\pi \bar V_0(t,x) = V(t,x) \;, \tag{3.12}$$

and the reconstruction operator $\mathscr{R}$ is such that $\mathscr{R}\hat\pi U = \mathscr{R}U$ for any modelled distribution U.

It then follows from (3.11) and the fact that we only consider admissible models that $u = \mathscr{R}V$ satisfies the distributional identity

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + u_0\delta_t + \mathscr{R}\mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta,\gamma}(V)\,\mathscr{D}V^\beta\,\mathscr{D}V^\gamma + h^\alpha(V)) + \eta^\alpha_r \;, \tag{3.13}$$

with η_r as above, where δ_t denotes the Dirac distribution centred at the origin in the time variable. Consider now the modelled distributions

$$X^\alpha_\varepsilon = \mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta\gamma}(\bar V_\varepsilon)\,\mathfrak{D}\bar V^\beta_\varepsilon\,\mathfrak{D}\bar V^\gamma_\varepsilon) \;, \qquad Y^\alpha = \mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta\gamma}(V)\,\mathfrak{D}V^\beta\,\mathfrak{D}V^\gamma) \;.$$

We make use of the following facts.

1. For any model of the type $\iota\mathbf{\Pi}$ with $\mathbf{\Pi} \in \mathscr{M}_0$, one has $\hat{\pi} X_0 = Y$ as a consequence of (3.12). In particular, for any such model one has $\mathscr{R} X_0 = \mathscr{R} Y$.
2. The map mapping the underlying model to $(\mathscr{R} X_\varepsilon, \mathscr{R} Y)$ is continuous from $\hat{\mathscr{M}}_\varepsilon$ to $\mathscr{D}'$. This follows from the results of [BCCH17] in virtually the same way as the continuity statement of Theorem 2.10.

Applying this to the specific choice of model given by $\hat{\mathbf{\Pi}}_\varepsilon$ as in Lemma 3.7, we conclude that for this choice one has $\mathscr{R} X_\varepsilon \to \mathscr{R} Y$ in probability. On the other hand, it follows from the results of [BCCH17] that there exists a function $G\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ depending on Γ and σ but not on h and such that one has the identity

$$\mathscr{R} X_\varepsilon = \mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\, \partial_x u_\varepsilon^\beta\, \partial_x u_\varepsilon^\gamma + G(u_\varepsilon, u)) \;.$$

(Indeed, it suffices to view $\hat{V}_\varepsilon = \mathscr{P} X_\varepsilon$ as an additional component of (3.11) and to note that the triangular structure of this system guarantees that the corresponding renormalisation term does not involve $\hat{V}_\varepsilon$ itself. The reason why the spatial derivatives of u and u_ε do not appear is the same as previously.)

Combining these facts together with (3.13), it follows that one has the almost sure identity

$$\eta_r^\alpha = \partial_t u^\alpha - \partial_x^2 u^\alpha - u_0 \delta_t - h^\alpha(u) - \lim_{\varepsilon\to 0} \mathbf{1}_{[0,\tau_r]}(\Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\, \partial_x u_\varepsilon^\beta\, \partial_x u_\varepsilon^\gamma + G(u_\varepsilon, u)) \;,$$

where the convergence of the last term takes place in probability in $\mathscr{D}'$ and therefore almost surely along a suitable sequence $\varepsilon \to 0$. The fact that we obtain (3.9) if we replace h by $\bar{h}$ in (3.11) is immediate from the construction. □

3.3 Equivariance of solutions

Recall the definition of $U_\varepsilon^{\mathrm{geo}}$ above and set $\bar{U}(\Gamma, h) = U_\varepsilon^{\mathrm{geo}}(\Gamma, 0, h)$, which is of course deterministic and independent of ε. Recall also that $\mathcal{S}_{\mathrm{geo}} \subset \mathcal{S}$ from Definition 3.2 denotes the space of all 'geometric' counterterms. The aim of this section is to show that the counterterm $C^{\mathrm{BPHZ}}_{\varepsilon,\mathrm{geo}}$ is "mostly" contained in $\mathcal{S}_{\mathrm{geo}}$ in the sense that its projection onto the complement of $\mathcal{S}_{\mathrm{geo}}$ converges to a finite limit.

For this, we first fix some complement $\mathcal{S}_{\mathrm{geo}}^\perp$ of $\mathcal{S}_{\mathrm{geo}}$ in $\mathcal{S}$ so that $\mathcal{S} = \mathcal{S}_{\mathrm{geo}} \oplus \mathcal{S}_{\mathrm{geo}}^\perp$. For definiteness, we could take the orthogonal complement with respect to the scalar product introduced in Section 2.4, but this is of no particular importance. We also fix a mollifier $\varrho \in \mathrm{Moll}$ and decompose the corresponding "geometric" BPHZ counterterm $C^{\mathrm{BPHZ}}_{\varepsilon,\mathrm{geo}}$ as $C^{\mathrm{BPHZ}}_{\varepsilon,\mathrm{geo}} = C^g_\varepsilon + C^c_\varepsilon$ with $C^g_\varepsilon \in \mathcal{S}_{\mathrm{geo}}$ and $C^c_\varepsilon \in \mathcal{S}_{\mathrm{geo}}^\perp$. The main result of this section is the following.

Proposition 3.9 *There exists* $v_{\mathrm{geo}} \in \mathcal{S}_{\mathrm{geo}}^\perp$ *such that* $\lim_{\varepsilon\to 0} C^c_\varepsilon = v_{\mathrm{geo}}$. *Furthermore,* v_{geo} *is independent of the choice of mollifier* ϱ.

Remark 3.10 Here and below, although the constants are independent of the choice of mollifier, they do in general depend on the choice of cutoff K of the heat kernel used in constructing our models.

Proof. Consider the decomposition $\mathcal{S} = \mathcal{S}^{(2)} \oplus \mathcal{S}^{(4)}$ according to how many noises appear in a given symbol. We then write $C^{\text{BPHZ}}_{\varepsilon,\text{geo},k}$ for the component of $C^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ in $\mathcal{S}^{(k)}$ and similarly for $C^g_{\varepsilon,k}$ and $C^c_{\varepsilon,k}$.

We first show that C^c_ε is bounded. Assume by contradiction that it is not so that, at least along a some subsequence $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} r_\varepsilon = +\infty\;, \quad r_\varepsilon = r_{\varepsilon,2} + r_{\varepsilon,4}\;, \quad r_{\varepsilon,k} = |C^c_{\varepsilon,k}|^{1/k}\;.$$

Set furthermore $\alpha_\varepsilon = 1/r_\varepsilon$. It is immediate that the pair $(\alpha_\varepsilon^2 C^c_{\varepsilon,2}, \alpha_\varepsilon^4 C^c_{\varepsilon,4}) \in (\mathcal{S}_{\text{geo}}^\perp)^2$ remains uniformly bounded as $\varepsilon \to 0$. It also remains uniformly bounded away from the origin since one has either $r_{\varepsilon,2} \geq r_{\varepsilon,4}$ in which case $|\alpha_\varepsilon^2 C^c_{\varepsilon,2}| \geq 1/4$ or $r_{\varepsilon,4} \geq r_{\varepsilon,2}$ in which case $|\alpha_\varepsilon^4 C^c_{\varepsilon,4}| \geq 1/16$. Modulo extracting a further subsequence, we can therefore assume that the pair converges to a non-degenerate limit $(\hat C_2, \hat C_4) \in (\mathcal{S}_{\text{geo}}^\perp)^2$. Note that we also have

$$v = \hat C_2 + \hat C_4 \neq 0\;,$$

since $\hat C_k \in \mathcal{S}^{(k)}$ and these spaces are transverse.

It follows from the definition of $\Upsilon_{\Gamma,\sigma}$ that, for $\tau \in \mathcal{S}^{(k)}$ and $r \in \mathbf{R}$, one has

$$\Upsilon_{\Gamma,r\sigma}\tau = r^k \Upsilon_{\Gamma,\sigma}\tau\;.$$

Combining this with Theorem 2.10 shows that, for every sequence $\alpha_\varepsilon \to 0$, we have the convergence in probability in $\mathscr{B}^a_\star$

$$\lim_{\varepsilon\to 0} U^{\text{geo}}_\varepsilon(\Gamma, \alpha_\varepsilon\sigma, h + \alpha_\varepsilon^2 \Upsilon_{\Gamma,\sigma} C^{\text{BPHZ}}_{\varepsilon,\text{geo},2} + \alpha_\varepsilon^4 \Upsilon_{\Gamma,\sigma} C^{\text{BPHZ}}_{\varepsilon,\text{geo},4}) = \bar U(\Gamma, h)\;.$$

With our particular choice of α_ε, this immediately implies that (along the subsequence chosen above)

$$\lim_{\varepsilon\to 0} U^{\text{geo}}_\varepsilon(\Gamma, \alpha_\varepsilon\sigma, \Upsilon_{\Gamma,\alpha_\varepsilon\sigma} C^g_\varepsilon) = \bar U(\Gamma, -\Upsilon_{\Gamma,\sigma} v)\;. \tag{3.14}$$

Given any diffeomorphism φ homotopic to the identity, we now conclude from (3.14) and the fact that the deterministic solution map is equivariant for the diffeomorphism group that

$$\begin{aligned}\bar U(\varphi\bullet\Gamma, -\varphi\bullet\Upsilon_{\Gamma,\sigma}v) = \varphi\bullet\bar U(\Gamma, -\Upsilon_{\Gamma,\sigma}v) &= \lim_{\varepsilon\to 0} U_\varepsilon(\varphi\bullet\Gamma, r_\varepsilon\varphi\bullet\sigma, r_\varepsilon^4 \Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma} C^g_\varepsilon)\\ &= \bar U(\varphi\bullet\Gamma, -\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma}v)\;.\end{aligned}$$

We conclude that $\varphi\bullet\Upsilon_{\Gamma,\sigma}v = \Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma}v$ as a consequence of the injectivity of $\bar U$ in its second argument which, since φ, Γ and σ were arbitrary, implies that $v \in \mathcal{S}_{\text{geo}}$ by definition. This however is in contradiction with the fact that $v \in \mathcal{S}_{\text{geo}}^\perp$ and $|v| \neq 0$.

This shows that C^c_ε is indeed bounded, so it remains to show that it can only have one accumulation point. Recall that, by the definition of U^{BPHZ} and Theorem 2.4, one has

$$U^{\text{BPHZ}}(\Gamma, \sigma, h) = \lim_{\varepsilon\to 0} U^{\text{geo}}_\varepsilon(\Gamma, \sigma, h + \Upsilon_{\Gamma,\sigma} C^{\text{BPHZ}}_{\varepsilon,\text{geo}})\;.$$

Acting again with an arbitrary diffeomorphism φ and applying Proposition 3.1, we conclude that

$$\begin{aligned}\varphi \bullet U^{\text{BPHZ}}(\Gamma,\sigma,0) &= \lim_{\varepsilon\to 0} U_\varepsilon^{\text{geo}}(\varphi\bullet\Gamma,\varphi\bullet\sigma,\varphi\bullet\Upsilon_{\Gamma,\sigma}C_{\varepsilon,\text{geo}}^{\text{BPHZ}})\\ &= \lim_{\varepsilon\to 0} U_\varepsilon^{\text{geo}}(\varphi\bullet\Gamma,\varphi\bullet\sigma,\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma}C_{\varepsilon,\text{geo}}^{\text{BPHZ}} + (\varphi\bullet\Upsilon_{\Gamma,\sigma}-\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma})C_\varepsilon^c)\;,\end{aligned}$$

which implies that the identity

$$\varphi\bullet U^{\text{BPHZ}}(\Gamma,\sigma,0) = U^{\text{BPHZ}}(\varphi\bullet\Gamma,\varphi\bullet\sigma,(\varphi\bullet\Upsilon_{\Gamma,\sigma}-\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma})v)\;, \tag{3.15}$$

holds for any accumulation point v of $\{C_\varepsilon^c\}_{\varepsilon\le 1}$. Since U^{BPHZ} is injective in its last argument by Theorem 3.5 and since this argument holds for any choice of Γ, σ and φ homotopic to the identity, we conclude that any two such accumulation points v and $\bar v$ necessarily satisfy

$$(\varphi\bullet\Upsilon_{\Gamma,\sigma}-\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma})(v-\bar v) = 0\;,$$

for all such Γ, σ and φ. This is precisely the definition of $\mathcal{S}_{\text{geo}}$, so that $v-\bar v\in\mathcal{S}_{\text{geo}}$, but since $v,\bar v\in\mathcal{S}_{\text{geo}}^\perp$ by construction, we conclude that $v=\bar v$ as claimed.

To show that v does not depend on the choice of mollifier either, we use (3.15) again in the same way. □

3.4 Itô isometry

We now show a statement analogous to Proposition 3.9, but this time regarding the "Itô isometry". Again, we fix a mollifier $\varrho\in$ Moll (symmetric, compactly supported, and non-anticipative) and decompose the corresponding "Itô" BPHZ counterterm $C_{\varepsilon,\text{Itô}}^{\text{BPHZ}}$ as $C_{\varepsilon,\text{Itô}}^{\text{BPHZ}} = C_\varepsilon^I + C_\varepsilon^c$ with $C_\varepsilon^I\in\mathcal{S}_{\text{Itô}}$ and $C_\varepsilon^c\in\mathcal{S}_{\text{Itô}}^\perp$ for some fixed choice of complement $\mathcal{S}_{\text{Itô}}^\perp$ of $\mathcal{S}_{\text{Itô}}$ in $\mathcal{S}$ so that $\mathcal{S}=\mathcal{S}_{\text{Itô}}\oplus\mathcal{S}_{\text{Itô}}^\perp$. The main result of this section is then the following.

Proposition 3.11 *There exists $v_{\text{Itô}}\in\mathcal{S}_{\text{Itô}}^\perp$ independent of the mollifier ϱ such that $\lim_{\varepsilon\to0} C_\varepsilon^c = v_{\text{Itô}}$.*

Proof. The proof is virtually identical to that of Proposition 3.9, so we only sketch it. One first shows that C_ε^c is bounded since otherwise one can again find $\alpha_\varepsilon\to 0$ and $v\in\mathcal{S}_{\text{Itô}}^\perp$ with $v\neq 0$ such that, along some subsequence,

$$\lim_{\varepsilon\to 0} U_\varepsilon^{\text{Itô}}(\Gamma,\alpha_\varepsilon\sigma,\Upsilon_{\Gamma,\alpha_\varepsilon\sigma}C_\varepsilon^I) = \bar U(\Gamma,-\Upsilon_{\Gamma,\sigma}v)\;. \tag{3.16}$$

Choosing any $\bar\sigma$ such that $\bar\sigma_i^\alpha\bar\sigma_i^\beta = \sigma_i^\alpha\sigma_i^\beta$, we similarly have

$$\lim_{\varepsilon\to 0} U_\varepsilon^{\text{Itô}}(\Gamma,\alpha_\varepsilon\bar\sigma,\Upsilon_{\Gamma,\alpha_\varepsilon\bar\sigma}C_\varepsilon^I) = \bar U(\Gamma,-\Upsilon_{\Gamma,\bar\sigma}v)\;. \tag{3.17}$$

On the other hand, one has $U_\varepsilon^{\text{Itô}}(\Gamma,\alpha_\varepsilon\sigma,\Upsilon_{\Gamma,\alpha_\varepsilon\sigma}C_\varepsilon^I) = U_\varepsilon^{\text{Itô}}(\Gamma,\alpha_\varepsilon\bar\sigma,\Upsilon_{\Gamma,\alpha_\varepsilon\bar\sigma}C_\varepsilon^I)$ by the second part of Proposition 3.1 and the definition of $\mathcal{S}_{\text{Itô}}$, so that one must have $\Upsilon_{\Gamma,\sigma}v = \Upsilon_{\Gamma,\bar\sigma}v$, which is in contradiction with the fact that $v\in\mathcal{S}_{\text{Itô}}^\perp$ does not vanish.

The argument that there can be only one accumulation point and that its value is independent of the mollifier is very similar, except that this time one obtains the identity

$$U^{\textsc{bphz}}(\Gamma,\sigma,0) = U^{\textsc{bphz}}(\Gamma,\bar\sigma,(\Upsilon_{\Gamma,\sigma} - \Upsilon_{\Gamma,\bar\sigma})v)\;,$$

from (3.16) and (3.17), so we omit it for conciseness. □

Corollary 3.12 *The assumption of Corollary 3.4 holds for every choice of mollifier* $\varrho \in \mathrm{Moll}$*. Furthermore, for* $i \in \{\mathrm{geo}, \mathrm{Itô}\}$*, if* $\mathcal{S}^{\mathrm{nice}} \subset \mathcal{S}$ *is a linear subspace such that both* $C^{\textsc{bphz}}_{\varepsilon,i} \in \mathcal{S}^{\mathrm{nice}}$ *for every choice of* $\varrho \in \mathrm{Moll}$*, then we can also choose* $\tau_i^{(\varepsilon)} \in \mathcal{S}^\star$*.*

Proof. Fix $i \in \{\mathrm{geo}, \mathrm{Itô}\}$ and write $\pi_i\colon \mathcal{S}^{\mathrm{nice}} \to \mathcal{S}^{\mathrm{nice}}_i$ for the projection associated to the decomposition $\mathcal{S}^{\mathrm{nice}} = \mathcal{S}^{\mathrm{nice}}_i \oplus (\mathcal{S}^{\mathrm{nice}}_i)^\perp$, where $\mathcal{S}^{\mathrm{nice}}_i = \mathcal{S}^{\mathrm{nice}} \cap \mathcal{S}_i$. It then suffices to set $\tau_i^{(\varepsilon)} = \pi_i C^{\textsc{bphz}}_{\varepsilon,i} + \tilde\tau_i$ for any fixed element $\tilde\tau_i \in \mathcal{S}^{\mathrm{nice}}_i$. □

Remark 3.13 We could of course have simply set $\tilde\tau_i = 0$, but leaving these two elements free will allow us to adjust them in the proof of Theorem 3.20 below in such a way that $\tau^{(\varepsilon)}_{\mathrm{geo}} - C^{\textsc{bphz}}_{\varepsilon,\mathrm{geo}}$ and $\tau^{(\varepsilon)}_{\mathrm{Itô}} - C^{\textsc{bphz}}_{\varepsilon,\mathrm{Itô}}$ converge to the same limit as $\varepsilon \to 0$ as already announced at the end of Section 3.1.

3.5 Combining both

Consider now the solution maps U^{geo} and $U^{\mathrm{Itô}}$ that are given by combining Corollaries 3.4 and 3.12. We already know from (3.4) and Propositions 3.9 and 3.11 that both notions of solution differ from the BPHZ solution by a fixed element in $\mathcal{S}$, so that there exists $\tau \in \mathcal{S}$ such that

$$U^{\mathrm{geo}}(\Gamma,\sigma,h) = U^{\mathrm{Itô}}(\Gamma,\sigma,h + \Upsilon_{\Gamma,\sigma}\tau)\;, \tag{3.18}$$

for every choice of Γ, σ, and h. Since $U^{\mathrm{Itô}}$ isn't covariant under changes of variables, there is however no a priori reason for $\Upsilon_{\Gamma,\sigma}\tau$ to transform like a vector field.

Take now a different collection of vector fields $\bar\sigma$ such that (3.2) holds. Then, one has

$$\begin{aligned}U^{\mathrm{geo}}(\Gamma,\bar\sigma,h) &= U^{\mathrm{Itô}}(\Gamma,\bar\sigma,h + \Upsilon_{\Gamma,\bar\sigma}\tau) = U^{\mathrm{Itô}}(\Gamma,\sigma,h + \Upsilon_{\Gamma,\bar\sigma}\tau)\\ &= U^{\mathrm{geo}}(\Gamma,\sigma,h + (\Upsilon_{\Gamma,\bar\sigma} - \Upsilon_{\Gamma,\sigma})\tau)\;.\end{aligned}$$

On the other hand, we know that, for any diffeomorphism φ of $\mathbf{R}^d$, if σ and $\bar\sigma$ satisfy (3.2), then one also has

$$(\varphi\bullet\bar\sigma_i)^\alpha(\varphi\bullet\bar\sigma_i)^\beta = (\varphi\bullet\sigma_i)^\alpha(\varphi\bullet\sigma_i)^\beta\;.$$

As a consequence,

$$\begin{aligned}\varphi\bullet U^{\mathrm{geo}}(\Gamma,\bar\sigma,h) &= U^{\mathrm{geo}}(\varphi\bullet\Gamma,\varphi\bullet\bar\sigma,\varphi\bullet h)\\ &= U^{\mathrm{geo}}(\varphi\bullet\Gamma,\varphi\bullet\sigma,\varphi\bullet h + (\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\bar\sigma} - \Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma})\tau)\;,\end{aligned}$$

as well as

$$\begin{aligned}\varphi \bullet U^{\mathrm{geo}}(\Gamma, \bar\sigma, h) &= \varphi \bullet U^{\mathrm{geo}}(\Gamma, \sigma, h + (\Upsilon_{\Gamma,\bar\sigma} - \Upsilon_{\Gamma,\sigma})\tau) \\ &= U^{\mathrm{geo}}(\varphi \bullet \Gamma, \varphi \bullet \sigma, \varphi \bullet h + \varphi \bullet (\Upsilon_{\Gamma,\bar\sigma} - \Upsilon_{\Gamma,\sigma})\tau)\ .\end{aligned}$$

Since U^{geo} is injective in its last argument, we conclude that τ is such that, for any choice of Γ, any pair $\sigma, \bar\sigma$ such that (3.2) holds, and any diffeomorphism φ of $\mathbf{R}^d$ homotopic to the identity, one has

$$\varphi \bullet (\Upsilon_{\Gamma,\bar\sigma} - \Upsilon_{\Gamma,\sigma})\tau = (\Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\bar\sigma} - \Upsilon_{\varphi\bullet\Gamma,\varphi\bullet\sigma})\tau\ . \tag{3.19}$$

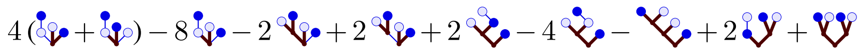
Denote by $\mathcal{S}_{\mathrm{both}} \subset \mathcal{S}$ the subspace of elements such that (3.19) holds for every choice of Γ, σ, $\bar\sigma$ and φ as above. With R denoting the Riemann curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z\ ,$$

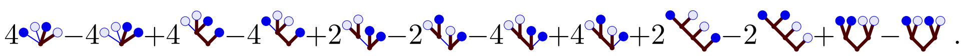
we define elements $\tau_\star, \tau_c \in \mathcal{S}$ by

$$\begin{aligned}\tau_\star &= R(\circ, \nabla_\bullet \circ - 2\nabla_\circ \bullet)\bullet\ , &(3.20)\\ \tau_c &= \nabla_\circ(R(\bullet,\circ)\bullet) - R(\nabla_\circ\bullet,\circ)\bullet - R(\bullet,\nabla_\circ\circ)\bullet - R(\bullet,\circ)\nabla_\circ\bullet\ . &(3.21)\end{aligned}$$

An explicit calculation shows that $8\tau_\star$ is equal to


(3.22)

and $8\tau_c$ is equal to


(3.23)

Lemma 3.14 *We have the identities*

$$\Upsilon^\alpha_{\Gamma,\sigma}\tau_\star = -R^\alpha_{\beta\gamma\eta}\, g^{\beta\zeta}\, (\nabla_\zeta g)^{\gamma\eta}\ , \qquad \Upsilon^\alpha_{\Gamma,\sigma}\tau_c = (\nabla_\zeta R)^\alpha_{\beta\gamma\eta}\, g^{\zeta\gamma} g^{\beta\eta}\ .$$

Proof. For the first identity, we start from the expression (3.20), which yields

$$\begin{aligned}\Upsilon^\alpha_{\Gamma,\sigma}\tau_\star &= R^\alpha_{\beta\gamma\eta}\Big(\sigma_j^\beta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_i^\eta - 2\sigma_i^\beta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_j^\eta\Big)\\ &= R^\alpha_{\beta\gamma\eta}\Big(-\sigma_j^\gamma(\nabla_{\sigma_i}\sigma_j)^\eta\sigma_i^\beta - \sigma_j^\eta(\nabla_{\sigma_i}\sigma_j)^\beta\sigma_i^\gamma - 2\sigma_i^\beta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_j^\eta\Big)\\ &= R^\alpha_{\beta\gamma\eta}\Big(-\sigma_j^\gamma(\nabla_{\sigma_i}\sigma_j)^\eta\sigma_i^\beta + \sigma_j^\eta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_i^\beta - 2\sigma_i^\beta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_j^\eta\Big)\\ &= R^\alpha_{\beta\gamma\eta}\Big(-\sigma_i^\beta(\nabla_{\sigma_i}\sigma_j)^\eta\sigma_j^\gamma - \sigma_i^\beta(\nabla_{\sigma_i}\sigma_j)^\gamma\sigma_j^\eta\Big)\\ &= -R^\alpha_{\beta\gamma\eta}\sigma_i^\beta(\nabla_{\sigma_i}g)^{\gamma\eta} = -R^\alpha_{\beta\gamma\eta}\sigma_i^\beta\sigma_i^\zeta(\nabla_\zeta g)^{\gamma\eta} = -R^\alpha_{\beta\gamma\eta}g^{\beta\zeta}(\nabla_\zeta g)^{\gamma\eta}\ ,\end{aligned}$$

where we first used the first Bianchi identity, then the antisymmetry of the curvature tensor and finally the definition of covariant derivative of g. Regarding the second identity, the definition of the covariant derivative of a tensor field implies that $\tau_c = (\nabla_\circ R)(\bullet,\circ)\bullet$, and the claim follows. ∎

Then, we have the following two identities.

Proposition 3.15 *With $\mathcal{S}_{\mathrm{both}}$, $\mathcal{S}_{\mathrm{Itô}}$ and $\mathcal{S}_{\mathrm{geo}}$ defined above, we have*

$$\mathcal{S}_{\mathrm{both}} = \mathcal{S}_{\mathrm{Itô}} + \mathcal{S}_{\mathrm{geo}}\;, \tag{3.24}$$

$$\mathcal{S}_{\mathrm{Itô}} \cap \mathcal{S}_{\mathrm{geo}} = \langle\{\tau_\star, \tau_c\}\rangle\;. \tag{3.25}$$

The proof of this statement is postponed to Section 6.2 below. Before we proceed, we argue that there is a natural way of "eliminating" the vector τ_c above. Indeed, denote by $\mathcal{S}^{\mathrm{nice}} \subset \mathcal{S}$ the subspace consisting of those elements $\tau \in \mathcal{S}$ such that, whenever Γ, σ and x are such that $\Gamma(x) = 0$ and $\partial\sigma(x) = 0$, one has $(\Upsilon_{\Gamma,\sigma}\tau)(x) = 0$. This space is easily characterised as follows.

Proposition 3.16 *The space $\mathcal{S}^{\mathrm{nice}}$ consists precisely of those elements $\tau \in \mathcal{S}$ such that*

$$\langle\;, \tau\rangle = \langle\;, \tau\rangle = \langle\;, \tau\rangle = 0\;. \tag{3.26}$$

Furthermore, one has $\perp \mathcal{S}_{\mathrm{Itô}}$, $\frac{1}{2} - - \frac{1}{2} \perp \mathcal{S}_{\mathrm{geo}}$, and $\langle\;, \tau_c\rangle \neq 0$.

Proof. The fact that $\mathcal{S}^{\mathrm{nice}}$ is characterised by (3.26) follows immediately from the non-degeneracy of the map $(\Gamma, \sigma, \tau) \mapsto \Upsilon_{\Gamma,\sigma}\tau$ encoded in Theorem 5.22 below, combined with the fact that and are the only two symbols in the list on page 16 which contain neither a node with exactly one incoming edge (corresponding to a factor $\partial\sigma$), nor a node with no incoming thin edge (corresponding to a factor Γ). The three trees appearing in (3.26) are the only inequivalent ways of pairing the noises of these two symbols.

The fact that $\perp \mathcal{S}_{\mathrm{Itô}}$ is a consequence of Proposition 6.4, the fact that $\frac{1}{2} - - \frac{1}{2} \perp \mathcal{S}_{\mathrm{geo}}$ is shown in Proposition 6.10 below, while the fact that $\langle\;, \tau_c\rangle \neq 0$ can be checked from the explicit expression of τ_c given in (3.23). □

Remark 3.17 An immediate consequence of this and of Proposition 6.10 is that $\mathcal{S}^{\mathrm{nice}} \cap \mathcal{S}_{\mathrm{geo}}$ is a subspace of $\mathcal{S}_{\mathrm{geo}}$ that is of codimension 2.

We also have the following elementary lemma.

Lemma 3.18 *Let B be a Banach space and let $V \subset B$ be a closed subspace of the form $V = \bigoplus_{i\in I} V_i$ for finitely many closed subspaces V_i. Let furthermore $W \subset B^*$ be a closed subspace of the dual space such that $W = \bigoplus_{i\in I} W_i$ where each closed subspace W_i satisfies $W_i \perp V_j$ for $i \neq j$. Then, one has*

$$V \cap W^\perp = \bigoplus_{i\in I} (V_i \cap W_i^\perp) = \bigoplus_{i\in I} (V_i \cap W^\perp)\;.$$

Proof. The proof is a simple exercise. □

Corollary 3.19 *Writing $\mathcal{S}^{\mathrm{nice}}_{\mathrm{Itô}} = \mathcal{S}_{\mathrm{Itô}} \cap \mathcal{S}^{\mathrm{nice}}$ and similarly for $\mathcal{S}^{\mathrm{nice}}_{\mathrm{geo}}$, etc, we have*

$$\mathcal{S}^{\mathrm{nice}}_{\mathrm{both}} = \mathcal{S}^{\mathrm{nice}}_{\mathrm{Itô}} + \mathcal{S}^{\mathrm{nice}}_{\mathrm{geo}}\;, \qquad \mathcal{S}^{\mathrm{nice}}_{\mathrm{Itô}} \cap \mathcal{S}^{\mathrm{nice}}_{\mathrm{geo}} = \langle\{\tau_\star\}\rangle\;.$$

Proof. Since $\mathcal{S}^{\text{nice}} = W^{\perp}$ for $W = \langle\{ , , \}\rangle$, it suffices to find decompositions of W and $V = \mathcal{S}_{\text{both}}$ satisfying the assumption of Lemma 3.18. We decompose W according to

$$W = \mathbf{R} \oplus \mathbf{R}(+ \tfrac{1}{2} - \tfrac{1}{2}) \oplus \mathbf{R}( ) = W_1 \oplus W_2 \oplus W_3\ .$$

We then choose $V_3 = \mathcal{S}_{\text{Itô}} \cap \mathcal{S}_{\text{geo}}$, $V_2 \subset \mathcal{S}_{\text{Itô}}$ any complement of V_3 in $\mathcal{S}_{\text{Itô}}$ which is orthogonal to (this is possible since does not annihilate $\tau_c \in V_3$), and finally V_1 any complement of V_3 in $\mathcal{S}_{\text{geo}}$ which is orthogonal to . Proposition 3.16 guarantees that these choices do satisfy the assumptions of Lemma 3.18. $\square$

We now have all the ingredients in place to show that it is possible to construct a solution map which satisfies both points 3 and 4 of Theorem 1.2 simultaneously.

Theorem 3.20 *For both $i \in \{\text{geo}, \text{Itô}\}$, there exists a choice of constants $\tilde\tau_i \in \mathcal{S}_i^{\text{nice}}$ independent of the mollifier $\varrho \in \text{Moll}$ such that, defining $\tau_i^{(\varepsilon)}$ as in the proof of Corollary 3.12, one has*

$$U^{\text{geo}} = U^{\text{Itô}}\ . \tag{3.27}$$

Furthermore, any two choices of $\tilde\tau_i$ having the same properties differ by a multiple of $\tau_\star$.

Proof. We first note that both $C^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ and $C^{\text{BPHZ}}_{\varepsilon,\text{Itô}}$ do indeed belong to $\mathcal{S}^{\text{nice}}$ irrespective of the choice of mollifier ϱ by Lemma 2.7.

In particular, we know that the element τ defined in (3.18) belongs to $\mathcal{S}^{\text{nice}}$ and is given by

$$\begin{aligned}\tau &= \lim_{\varepsilon\to 0}\big(\tau^{(\varepsilon)}_{\text{geo}} - C^{\text{BPHZ}}_{\varepsilon,\text{geo}} - \tau^{(\varepsilon)}_{\text{Itô}} + C^{\text{BPHZ}}_{\varepsilon,\text{geo}}\big) \\ &= \tilde\tau_{\text{geo}} - v_{\text{geo}} - \tilde\tau_{\text{Itô}} + v_{\text{Itô}}\ ,\end{aligned}$$

for $v_i \in \mathcal{S}_i^{\perp}$ as in Propositions 3.9 and 3.11 and $\tilde\tau_i \in \mathcal{S}_i^{\text{nice}}$ as in the proof of Corollary 3.12. We know furthermore that $\tau \in \mathcal{S}^{\text{nice}}_{\text{both}}$ by (3.19), so that by Corollary 3.19 we can choose $\tilde\tau_{\text{geo}} \in \mathcal{S}^{\text{nice}}_{\text{geo}}$ and $\tilde\tau_{\text{Itô}} \in \mathcal{S}^{\text{nice}}_{\text{Itô}}$ in such a way that $\tau = 0$, as desired. The second statement is immediate from the second part of Corollary 3.19 and the injectivity of the solution maps in their last argument. $\square$

4 Proof of main results

We now have almost all of the ingredients in place to prove our main result. Before we turn to it however, we need one more result on the behaviour of the BPHZ renormalisation constants.

4.1 Convergence of constants

Throughout this section, we write C_ε for the BPHZ character $C^{\text{BPHZ}}_{\varepsilon,\text{geo}}$ from Proposition 2.1. We interpret this character as an element of $\mathcal{S}^{\text{nice}}$ in such a way that the counterterm is given by $\Upsilon_{\Gamma,\sigma} C_\varepsilon$. Recall also the definition of $\tau_\star$ from (3.20). The main result in this section is as follows.

Theorem 4.1 *There exists an element $\tau_0 \in \mathcal{S}^{\text{nice}}$ and a constant $\bar c > 0$ such that*

$$\lim_{\varepsilon\to 0}\Big(C_\varepsilon + \frac{\bar c}{\varepsilon}\nabla_{\circ\circ} - \frac{\log\varepsilon}{4\sqrt{3}\pi}\tau_\star\Big) = \tau_0\;.$$

Proof. By Proposition 3.9, there exists $v_{\text{geo}} \in \mathcal{S}^\perp_{\text{geo}} \subset \mathcal{S}^{\text{nice}}$ such that the distance between $C_\varepsilon - v_{\text{geo}}$ and the 13-dimensional 'geometric' subspace $\mathcal{V}^{\text{nice}} = \mathcal{S}^{\text{nice}}_{\text{geo}}$, see Corollary 3.19 and Proposition 6.10 below, converges to 0 as $\varepsilon \to 0$. Using (3.22), we have that $\langle \tau, \nabla_{\circ\circ}\rangle = \langle\tau,\tau_\star\rangle = 0$ for

$$\tau \in \Big\{\;,\;,\;,\;,\;,\;,\;,\;,\;,\; - \;,\;\Big\}\;, \tag{4.1}$$

and by Proposition 6.10, these form 11 linearly independent linear functionals on $\mathcal{S}^{\text{nice}}_{\text{geo}}$. In order to complete the proof, it therefore suffices to show that $\langle C_\varepsilon, \tau\rangle$ converges to a finite limit as $\varepsilon \to 0$ for every τ as in (4.1) and, since $\langle\;, \nabla_{\circ\circ}\rangle = \langle 8\;, \tau_\star\rangle = 1$, that the limits

$$\lim_{\varepsilon\to0}\Big(\langle C_\varepsilon, \;\rangle + \frac{\bar c}{\varepsilon}\Big)\;,\qquad \lim_{\varepsilon\to0}\Big(\langle C_\varepsilon, \;\rangle - \frac{\log\varepsilon}{32\sqrt{3}\pi}\Big) \tag{4.2}$$

exist and are finite.

By Lemma 2.7, $\langle C_\varepsilon, \tau\rangle = 0$ for $\tau \in \{\;,\;,\;,\;,\;,\;\}$. In [HP15], it was furthermore shown that $\langle C_\varepsilon,\tau\rangle$ converges to finite limits for $\tau \in \{\;,\;,\;,\;\}$, while Lemma 4.3 below shows that $\langle C_\varepsilon, \; - \;\rangle$ converges to a finite limit.

The fact that the first limit in (4.2) actually vanishes for

$$\bar c = \int_{\mathbf{R}^2} ((\partial_x P \star \varrho)(t,x))^2\,dt\,dx$$

is a simple calculation exploiting the scale invariance of the hear kernel. The fact that the second limit in (4.2) is also finite is the content of Lemma 4.4 below, which concludes the proof. $\square$

Remark 4.2 Note that there is a sign error in [HS17] before Eq. 3.6: the factors $\log\varepsilon$ should read $|\log\varepsilon|$, which then makes it consistent with (4.2).

We now show that the BPHZ renormalisation constant yields a finite limit on $\; - \;$ and compute the prefactor of the log-divergence.

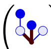
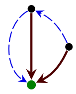
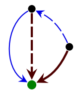
Lemma 4.3 *For every mollifier ϱ as above, there exists a constant c such that*

$$\lim_{\varepsilon\to0} C^{\text{BPHZ}}_\varepsilon\big(\; - \;\big) = c\;.$$

Proof. As in [HQ18, Sec. 6.3], it will be convenient to perform a change of variables and to write

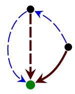
$$K_{\varepsilon,\varrho}(z) = (\varrho * \mathcal{S}_\varepsilon^{(1)} K)(z)\;, \qquad K_\varepsilon(z) = (\mathcal{S}_\varepsilon^{(1)} K)(z)\;,$$

where $(\mathcal{S}_\varepsilon^{(\alpha)} K)(t,x) = \varepsilon^\alpha K(\varepsilon^2 t, \varepsilon x)$. Writing $\dashrightarrow$ for $K_{\varepsilon,\varrho}$, $\dashrightarrow$ for $K'_{\varepsilon,\varrho}$, and the 'plain' versions of these arrows for K_ε and K'_ε, we then have

$C_\varepsilon^{\text{BPHZ}}\Big($$\Big) =$ , $\qquad C_\varepsilon^{\text{BPHZ}}\Big($$\Big) =$ 

At this point, we note that with the notations of [HQ18, Sec. 6.3], $K_{\varepsilon,\varrho}$, $K'_{\varepsilon,\varrho}$, K_ε, and K_ε converge to finite limits in $\mathcal{B}_{1-\kappa,0}$, $\mathcal{B}_{2-\kappa,0}$, $\mathcal{B}_{1-\kappa,1+\kappa}$ and $\mathcal{B}_{2-\kappa,2+\kappa}$ respectively. Furthermore, $K_{\varepsilon,\varrho} - K_\varepsilon$ and $K'_{\varepsilon,\varrho} - K'_\varepsilon$ converge to finite limits in $\mathcal{B}_{2-\kappa,1+\kappa}$ and $\mathcal{B}_{3-\kappa,2+\kappa}$.

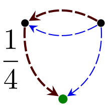
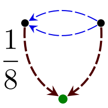
Recall also that the product is continuous from $\mathcal{B}_{\alpha,\beta} \times \mathcal{B}_{\bar\alpha,\bar\beta}$ into $\mathcal{B}_{\alpha+\bar\alpha,\beta+\bar\beta}$ and that the functional $K \mapsto \int_{\mathbf{R}^2} K(z)\,dz$ is continuous on $\mathcal{B}_{\alpha,\beta}$ if $\alpha > 3$ and $\beta < 3$. Setting

$c_\varepsilon^\varrho =$ ,

we conclude from these facts and from [HQ18, Lem. 6.9] that both $C_\varepsilon^{\text{BPHZ}}(\cdot) - c_\varepsilon^\varrho$ and $C_\varepsilon^{\text{BPHZ}}(\cdot) - c_\varepsilon^\varrho$ converge to finite limits, which implies the claim. $\square$

Lemma 4.4 *The second limit in (4.2) is finite.*

Proof. It is shown in [HQ18, Eq. 6.31] that $\langle C_\varepsilon, \cdot\rangle$ (which is called $C_2^{(\varepsilon)}$ in that article) differs by a converging term from

$L_\varepsilon = \frac{1}{4}$  $= -\frac{1}{8}$ ,

where we performed an integration by parts on the top right integration variable. Using [HQ18, Lem 6.12], we see that this differs by a converging quantity from

$$-\frac{1}{16}\int_{\mathbf{R}^2} K_{\varepsilon,\varrho}^3(t,x)\,dt\,dx\;. \tag{4.3}$$

Using the identity $\int_{\mathbf{R}} P^3(t,x)\,dx = 1/(4\sqrt{3}\pi t)$ and the scaling properties of the heat kernel, a simple asymptotic analysis shows that $\int_{\mathbf{R}^2} K_{\varepsilon,\varrho}^3(t,x)\,dx$ differs from $\mathbf{1}_{[\varepsilon^2,1]}/(4\sqrt{3}\pi t)$ by a term whose integral converges to a finite limit. Inserting this into (4.3) and performing the integration over t yields the desired result. $\square$

4.2 Proof of the main theorem

We are now in a position to combine all of our results.

Proof of Theorem 1.2. Recall first the definition of $\tau_0 \in \mathcal{S}^{\text{nice}}$ from Theorem 4.1 and note that the space $\mathscr{V}^{\text{nice}}$ mentioned in the introduction is nothing but the space $\mathcal{S}^{\text{nice}}_{\text{geo}}$. Since $\Upsilon_{\Gamma,\sigma}\tau_\star = H_{\Gamma,\sigma}$, it follows that claims 1, 3 and 4 of Theorem 1.2 hold as long as $c \in \mathcal{S}^{\text{nice}}_{\text{geo}}$ and $\hat{c} \in \mathbf{R}$ are chosen in such a way that

$$c + \hat{c}\tau_\star = \lim_{\varepsilon\to 0}\Big(\tau^{(\varepsilon)}_{\text{geo}} + \frac{\bar{c}}{\varepsilon}\nabla_{\circ}\circ - \frac{\log\varepsilon}{4\sqrt{3}\pi}\tau_\star\Big) ,$$

where $\tau^{(\varepsilon)}_{\text{geo}}$ is as in Corollary 3.12 and Theorem 3.20 and the convergence of the right hand side is guaranteed by Theorem 4.1. The uniqueness claim is then the same as the uniqueness claim modulo $\tau_\star$ that is found in Theorem 3.20. The fact that (1.11) holds for arbitrary diffeomorphisms (and not just those homotopic to the identity) follows from the fact that the counterterms in $\mathcal{S}_{\text{geo}}$ are obtained by taking multiple covariant derivatives of the vector fields σ_i, see Remark 6.16 below.

Claim 2 is an immediate consequence of the fact that the driving noise is white and the stability of the limit with respect to perturbations in the initial condition. More details can be found for example in [Hai14, Sec. 7.3].

To show that claim 5 holds, note first that, as a consequence of Lemma 3.6, the solutions $U^{\text{BPHZ}}(0,\sigma,h)$ given by the BPHZ model do coincide with the classical Itô solutions to (1.13). It therefore remains to show that $U^{\text{BPHZ}}(0,\sigma,h) = U^{\text{Itô}}(0,\sigma,h)$. This follows immediately from the fact that $U^{\text{BPHZ}}(0,\sigma,h) = U^{\text{Itô}}(0,\sigma,h+\Upsilon_{0,\sigma}\tau)$ for some element $\tau \in \mathcal{S}_{\text{Itô}}$. Now $\Upsilon_{0,\sigma}\tau$ vanishes unless the symbol τ contains only nodes of type $\circ$ and no node of type $\odot$, while $\mathcal{S}_{\text{Itô}}$ is orthogonal to any such symbol by Proposition 6.11 and the second statement of Theorem 6.15.

Finally, the last statement follows as in [Hai11, Cor. 3.11], noting that since we are in the situation $\Gamma = 0$ and σ constant, $\Upsilon_{\Gamma,\sigma}$ vanishes identically on all of $\mathcal{S}$, so that in this case our notion of solution coincides simply with the limit of (1.5) as $\varepsilon \to 0$. $\square$

4.3 Path integrals and a conjecture

Let again $\mathscr{M}$ be a Riemannian manifold with inverse metric tensor g, let A be a smooth vector field on $\mathscr{M}$, and let μ be the loop measure on $\mathscr{C}^a(S^1,\mathscr{M})$ determined by the diffusion with generator

$$Lf = \frac{1}{2}\Delta f + df(A) ,$$

where Δ is the Laplace-Beltrami operator on $\mathscr{M}$. We assume furthermore that the corresponding diffusion process does not explode in finite time and is such that $x \mapsto p_1(x,x)$ is integrable on $\mathscr{M}$. It is a simple exercise in stochastic calculus to verify that the measure μ is equivalent to the measure μ_0 constructed like μ but with

$A = 0$. Its Radon-Nikodym derivative is given by

$$\exp\Big(\int_0^1 \langle A(u), \circ du(x)\rangle_{g(u)} - \frac{1}{2}\int_0^1 (|A(u)|^2_{g(u)} + \operatorname{div} A(u))\, dx\Big) ,$$

where $\circ$ denotes Stratonovich integration. In view of this, it is then natural to conjecture the following (see [Hai11] for a statement and proof in the Euclidean case).

Conjecture 4.5 In the above situation, let σ_i be any collection of smooth vector fields such that $g = \sigma_i \otimes \sigma_i$ and let $\Gamma^\alpha_{\beta\gamma}$ be the Christoffel symbols for the Levi-Civita connection on $\mathscr{M}$. Then, there exists a universal constant c such that the unique process given by canonical solution of Theorem 1.5 for the stochastic PDE

$$\partial_t u = \nabla_{\partial_x u}\partial_x u + ((dA^\flat)(\partial_x u))^\sharp - \nabla_A A - \frac{1}{2}\nabla \operatorname{div} A + c\nabla R(u) + \sqrt{2}\sigma_i(u)\,\xi_i , \tag{4.4}$$

where R denotes the scalar curvature of $\mathscr{M}$, exists for all times and has μ as its unique invariant measure. Comparing with (1.5), the coefficients there become here $h = -\nabla_A A - \frac{1}{2}\nabla \operatorname{div} A + c\nabla R$, while K is the tensor field such that $K\partial_x u = ((dA^\flat)(\partial_x u))^\sharp$.

Remark 4.6 If we can find a sequence of approximations u^ε to (4.4) such that the corresponding invariant measures μ_ε converge to μ in a sufficiently strong sense (all moments in $\mathscr{C}^a$ bounded), then the claim follows as in [HM18a, Cor. 1.3]. In particular, uniqueness of the invariant measure (restricted to any given homotopy class) then follows by combining [HM18b, Thm 4.8] with the Stroock-Varadhan support theorem [SV72] which shows that the measure μ has full support.

It is not a priori obvious what the value of the constant c appearing in Conjecture 4.5 should be. One would of course be tempted to conjecture that $c = 0$, but there is no good a priori reason why this would be the case. After all, the only reason why we obtain a canonical notion of solution is that we restrict a priori our renormalisation constants to the subspace $\mathscr{S}^{\text{nice}} \subset \mathscr{S}$ which is chosen in a rather ad hoc manner: it does not arise from any symmetry consideration but from a remark on the structure of the BPHZ renormalisation constants.

Parsing the literature, there is consensus on the fact that if one wishes to represent Brownian motion on $\mathscr{M}$ by a path integral, then it should indeed be of the form

$$\exp\Big(-\frac{1}{2}\int_0^1 |\partial_x u|^2_{g(u)}\, dx + c\int_0^1 R(u)\, dx\Big) ,$$

(for which (4.4) with $h = 0$ would formally be the natural Langevin dynamic) but there is no consensus on the value of c. For example, the value $c = \frac{1}{12}$ would be consistent with the Onsager-Machlup functional appearing in the main theorem of [TW81] (see also [Gra77] for example). The result [AD99] shows that for two rather natural choices of the Riemannian volume form for a piecewise geodesic discretisation of path space, one obtains either $c = 0$ or $c = \frac{1}{6}$. The value $c = \frac{1}{6}$ also appears in [Che72, Eq. 15] (it may look like it has the opposite sign but if we interpret their statement in our context, it states that the invariant measure for (4.4) with $c = 0$ and $h = 0$ is given by the Brownian loop measure weighted by $\exp(-\frac{1}{6}\int R(u)\,dx)$ which is equivalent to the statement that our conjecture holds with $c = \frac{1}{6}$). The value $c = \frac{1}{8}$ does appear in [DeW92, Eq. 6.5.25] (but the same author also makes a case for $c = -\frac{1}{12}$ in [Dew57, Eq. 7.12]), while $c = -\frac{1}{8}$ appears in [Dek80, Eq. 4.9]. With our particular choice of normalisation, we have the following conjecture.

Conjecture 4.7 One has $c = \frac{1}{8}$ in (4.4).

The remainder of this section is devoted to an argument in favour of this conjecture. At this stage, this is of heuristic nature, but we believe that it could in principle be made rigorous by combining the results of [EH17] with a suitable discrete version of the results of [CH16]. We take as our starting point the result of [AD99] which shows that, approximating a path in $\mathscr{M}$ with a sequence of $N = 1/\varepsilon$ points, the sequence of measures μ_n on $\mathscr{M}^N$ with density (with respect to the product measure) proportional to

$$\exp\Big(-\frac{1}{2}\sum_k \frac{d(u_k, u_{k+1})^2}{\varepsilon} + \frac{1}{6}\sum_k R(u_k)\,\varepsilon\Big)\;,$$

where R denotes the scalar curvature of $\mathscr{M}$, converge to μ as $\varepsilon \to 0$. Considering the corresponding Langevin dynamic, we find that the measure μ_n is invariant for the system of coupled SDEs on $\mathscr{M}^N$ given by

$$du_k = \frac{\exp_{u_k}^{-1}(u_{k+1}) + \exp_{u_k}^{-1}(u_{k-1})}{\varepsilon^2}\,dt + \frac{1}{6}\nabla R(u_k)\,dt + \sqrt{\frac{2}{\varepsilon}}\sigma_i(u)\circ dW_{i,k}\;, \tag{4.5}$$

where the $W_{i,k}$ are i.i.d. standard Wiener processes and the σ_i are a collection of vector fields as above satisfying furthermore the property $\nabla_{\sigma_i}\sigma_i = 0$. (We also use the convention that the index of u is interpreted modulo N.) We now fix a coordinate chart and write $\delta_k^{\pm} u^\alpha = u_{k\pm1}^\alpha - u_k^\alpha$. We also define a valuation $\hat\Upsilon_\Gamma(u,v)$ in the same way as $\Upsilon_{\Gamma,\sigma}$, but it acts on symbols with edges of type $\square$ rather than edges of type $\circ_i$ and it replaces each instance of $\square$ with v. We furthermore restrict this to "saturated" symbols in the sense that vertices without noise edge have two incoming thick edges, so that each instance of $\odot$ is replaced by a suitable derivative of 2Γ. For example, one has

$$\hat\Upsilon_\Gamma^\alpha(u,v)\,\vee = 2\Gamma_{\beta\gamma}^\alpha(u)v^\beta v^\gamma\;.$$

(This of course only makes sense for symbols which have $\square$ as leaves.) A lengthy but ultimately not terribly interesting calculation shows that, in local coordinates, the first term in this equation can be approximated to fourth order in δu by

$$\exp_{u_k}^{-1}(u_{k+1}) + \exp_{u_k}^{-1}(u_{k-1}) \approx (\hat\Upsilon_\Gamma(u_k, \delta_k^+ u) + \hat\Upsilon_\Gamma(u_k, \delta_k^- u))\tau\;, \tag{4.6}$$

where

$$\tau = \quad + \frac{1}{4} \quad + \frac{1}{24} \quad + \frac{1}{12} \quad + \frac{1}{96} \quad + \frac{1}{48} \quad + \frac{1}{48} \quad + \frac{1}{192} \quad . \tag{4.7}$$

In terms of powercounting, all of these terms should disappear as $\varepsilon \to 0$ except for the first two, which yield the natural discrete approximation to $\nabla_{\partial_x u} \partial_x u$. (The factor $\frac{1}{2}$ that appears in front of is cancelled out by the fact that every term is doubled in the right hand side of (4.6).) As a consequence, one would expect that, in local coordinates, the BPHZ renormalised solution to (4.5) converges as $\varepsilon \to 0$ to the BPHZ renormalised solution to (4.4) (with $h = 0$). On the other hand, we know that the BPHZ solution to (4.4) in any fixed chart differs from its canonical solution by a counterterm $\Upsilon_{\Gamma,\sigma}\tau_0$ with $\tau_0 \in \mathcal{S}^{\text{nice}}$, which in particular does *not* include any term proportional to $\Upsilon_{\Gamma,\sigma}$ or $\Upsilon_{\Gamma,\sigma}$. Since on the other hand the expression for ∇R *does* contain the term $\Upsilon_{\Gamma,\sigma}(\; - \;)$, it suffices to focus on these two terms in order to be able to conjecture the value of c.

Denote now by $\tilde{u}_i$ the (stationary, mean 0) solution to the linearised equation, namely

$$d\tilde{u}_{i,k} = \frac{\delta_k^+ \tilde{u}_i + \delta_k^- \tilde{u}_i}{\varepsilon^2} + \sqrt{\frac{2}{\varepsilon}} dW_{i,k} \; . \tag{4.8}$$

The $\tilde{u}_i$ are independent, Gaussian, and satisfy

$$\varepsilon^{-1} \mathbf{E} |\delta_k^\pm \tilde{u}_i|^2 \to 1 \;, \qquad \varepsilon^{-1} \mathbf{E} u_{i,k} \delta_k^\pm \tilde{u}_i \to -\frac{1}{2} \; . \tag{4.9}$$

(The first limit comes from the fact that the invariant measure of (4.8) converges to the Brownian loop measure. The second limit follows from the first by exploiting the $k \mapsto k+1$ and $k \mapsto N-k$ symmetries.)

A formal expansion then suggests that counterterms of the type $\Upsilon_{\Gamma,\sigma}$ and $\Upsilon_{\Gamma,\sigma}$ can be generated by the terms , and appearing in τ. The counterterm generated by is given by

$$\frac{1}{4} \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{4} \cdot \Upsilon_{\Gamma,\sigma} \quad = \frac{1}{8} \Upsilon_{\Gamma,\sigma} \quad .$$

The factor $\frac{1}{4}$ comes from (4.7), the factor 2 from the fact that each of these terms is doubled in (4.6), the factor $\frac{1}{2}$ from the second order Taylor expansion, the factor 2 from the two ways of pairing up the two extra factors of u with the factors $\delta^\pm u$, and the final factor $\frac{1}{4}$ from the second limit in (4.9). Similarly, the term generates a counterterm given by

$$\frac{1}{12} \cdot 2 \cdot \Big(-\frac{1}{2}\Big) \cdot \Big(\Upsilon_{\Gamma,\sigma} \quad + 2 \Upsilon_{\Gamma,\sigma} \quad \Big) = -\frac{1}{12} \Upsilon_{\Gamma,\sigma} \quad - \frac{1}{6} \Upsilon_{\Gamma,\sigma} \quad ,$$

while the term generates the counterterm

$$\frac{1}{48} \cdot 2 \cdot \Big(\Upsilon_{\Gamma,\sigma} \quad + 2 \Upsilon_{\Gamma,\sigma} \quad \Big) = \frac{1}{24} \Upsilon_{\Gamma,\sigma} \quad + \frac{1}{12} \Upsilon_{\Gamma,\sigma} \quad .$$

Combining all of these terms with the term $\frac{1}{6} \Upsilon_{\Gamma,\sigma}(\; - \;)$ coming from the curvature term in (4.5) (recall that τ_c is given by (3.23) and that $\Upsilon_{\Gamma,\sigma} \tau_c = -2 \nabla R$ by Remark 1.13 and Lemma 3.14) yields $\frac{1}{8} \Upsilon_{\Gamma,\sigma}(\; - \;)$, allowing us to conclude.

4.4 Loops on a manifold

The aim of this section is to give a proof of Theorem 1.5. Throughout this section, we fix some $\mathbf{R}^d$, as well as a Riemannian manifold $\mathcal{M}$ that is smoothly and isometrically embedded into $\mathbf{R}^d$. We then recall the following fact about such a setup.

Lemma 4.8 *Let $\pi \colon \mathbf{R}^d \to \mathcal{M}$ be a smooth function such that $\pi\restriction\mathcal{M} = \mathrm{id}$ and such that, for every $p \in \mathcal{M}$ and $v \in (T_p\mathcal{M})^\perp$ one has $D_v\pi(p) = 0$. Let furthermore $\bar X$ and $\bar Y$ be smooth vector fields on $\mathbf{R}^d$ and write X and Y for their restrictions to $\mathcal{M}$. Define*

$$(\bar\nabla_{\bar X}\,\bar Y)^\alpha(x) = \bar X^\beta(x)\,\partial_\beta \bar Y^\alpha(x) - \partial^2_{\beta\gamma}\pi^\alpha(x)\,\bar X^\beta(x)\,\bar Y^\beta(x)\;.$$

Then, for every $x \in \mathcal{M}$ one has $(\bar\nabla_{\bar X}\bar Y)(x) = (\nabla_X Y)(x) \in T_x\mathcal{M}$.

Proof. Since $\mathbf{R}^d$ is flat, Lie derivatives and covariant derivatives coincide. Furthermore, our assumption guarantees that, for every $x \in \mathcal{M}$, $D\pi(x)$ is the orthogonal projection onto $T_x\mathcal{M}$, so that, for $x \in \mathcal{M}$, one has

$$(\nabla_X Y)^\alpha(x) = \partial_\beta\pi^\alpha(x)\,\bar X^\gamma(x)\,\partial_\gamma \bar Y^\beta(x)\;. \tag{4.10}$$

(See for example [Pet06, Prop. 5.13 (4)].) On the other hand, since $\bar Y(x) \in T_x\mathcal{M}$ for $x \in \mathcal{M}$, one has the identity

$$\bar Y^\alpha(x) = \partial_\beta\pi^\alpha(x)\,\bar Y^\beta(x)\;, \qquad \forall x \in \mathcal{M}\;.$$

Differentiating this identity in the direction $\bar X(x) \in T_x\mathcal{M}$, we obtain on $\mathcal{M}$

$$\bar X^\gamma \partial_\gamma \bar Y^\alpha(x) = \partial^2_{\beta,\gamma}\pi^\alpha(x)\,\bar X^\gamma(x)\,\bar Y^\beta(x) + \partial_\beta\pi^\alpha(x)\,\bar X^\gamma\,\partial_\gamma \bar Y^\beta(x)\;.$$

Combining this with (4.10), the claim follows. □

A natural collection of vector fields σ_i is now given by setting

$$\sigma_i^\alpha(x) = \partial_i\pi^\alpha(x)\;. \tag{4.11}$$

It is clear that, for all $x \in \mathcal{M}$ one has $\sigma_i(x) \in T_x\mathcal{M}$ as a consequence of the fact that π maps x to itself and maps any neighbourhood of x to $\mathcal{M}$. With this definition, we have the following.

Lemma 4.9 *In the above setting, with π as in Lemma 4.8 and with σ as in (4.11) viewed as vector fields on $\mathcal{M}$, one has $\sum_{i=1}^d \sigma_i \otimes \sigma_i = g$ and $\sum_{i=1}^d \nabla_{\sigma_i}\sigma_i = 0$.*

Proof. Since, for $x \in \mathcal{M}$, $D\pi(x)$ equals the *orthogonal* projection P_x onto $T_x\mathcal{M}$, one has the identity

$$\sum_{i=1}^d \sigma_i^\alpha(x)\,\sigma_i^\beta(x) = P_x^{\alpha\beta} = g^{\alpha\beta}(x)\;, \tag{4.12}$$

where the first identity is nothing but the identity $P_x P_x^* = P_x$ which holds for any orthogonal projection, while the second identity is the definition of an isometric embedding of $\mathscr{M}$ into $\mathbf{R}^d$. By Lemma 4.8, one then has

$$\begin{aligned}\sum_{i=1}^d (\nabla_{\sigma_i}\sigma_i)^\alpha(x) &= \sum_{i=1}^d (\partial_i\pi^\beta\,\partial_\beta\partial_i\pi^\alpha - \partial^2_{\beta\gamma}\pi^\alpha\,\partial_i\pi^\beta\,\partial_i\pi^\gamma)(x)\\ &= P_x^{\beta\gamma}\,\partial^2_{\beta\gamma}\pi(x) - \partial^2_{\beta\gamma}\pi^\alpha(x)(P_xP_x^*)^{\beta\gamma} = 0\;,\end{aligned}$$

where we made use of (4.12). □

We now have all the ingredients in place to give a proof of Theorem 1.5.

Proof of Theorem 1.5. By Nash's embedding theorem [Nas56], it is possible to find an isometric embedding of $\mathscr{M}$ in $\mathbf{R}^d$ for d sufficiently large. We fix such an embedding (hence view $\mathscr{M}$ as a subset of $\mathbf{R}^d$) and fix a map $\pi\colon \mathbf{R}^d \to \mathscr{M}$ as in Lemma 4.8. For example, we can choose $\pi(y) = \operatorname{arginf}\{|x-y| \,:\, x\in\mathscr{M}\}$ in a sufficiently small neighbourhood of $\mathscr{M}$ and then extend this in an arbitrary smooth way to all of $\mathbf{R}^d$. We also choose σ as in (4.11).

For any $\varrho \in$ Moll and $\varepsilon > 0$, we consider the solution $u_\varepsilon\colon \mathbf{R}_+\times S^1 \to \mathbf{R}^d$ to

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha - \partial^2_{\beta\gamma}\pi^\alpha(u_\varepsilon)\,\partial_x u_\varepsilon^\beta\,\partial_x u_\varepsilon^\gamma + h^\alpha(u_\varepsilon) + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon + V^\alpha_{\varrho,\sigma}(u_\varepsilon)\;. \tag{4.13}$$

It follows from Lemma 4.8 that, provided that $u_\varepsilon(t,x)\in\mathscr{M}$, the right hand side of (4.13) coincides with that of (1.14).

It is then a standard fact (see for example [LT91, Lem. 3.1] or the original article [ES64]) that, provided that the vector field h is tangent to $\mathscr{M}$, solutions to (4.13) that start on $\mathscr{M}$ stay on $\mathscr{M}$ for all subsequent times and solve (1.14). Furthermore, for any chart $\mathscr{U}$ of $\mathscr{M}$, if we take an initial condition that is entirely contained in $\mathscr{U}$ then, for as long as this remains the case, the solution satisfies (1.5), with $\Gamma^\alpha_{\beta\gamma}$ given by the Christoffel symbols of the Levi-Civita connection for the Riemannian metric of $\mathscr{M}$ in the chart $\mathscr{U}$.

On the other hand, (4.13) is itself of the form (1.5), so that we can apply Theorem 1.2. In this particular case however, one has $\nabla_\circ\circ = 0$ on $\mathscr{M}$ by Lemma 4.9. One also has $H_{\Gamma,\sigma} = 0$ on $\mathscr{M}$ since the connection given by (4.10) is the Levi-Civita connection and, by (1.9), $H_{\Gamma,\sigma}$ is built from the covariant derivative of the metric, which vanishes in that case. These properties may fail to hold on all of $\mathbf{R}^d$ in general, but since we restrict ourselves to solutions that start and remain on $\mathscr{M}$ at all times, this is of no concern to us. This shows that the 'canonical family' exhibited in Theorem 1.2 is really a 'canonical solution', implying both the convergence and the uniqueness claim made in the statement of Theorem 1.5. The last statement follows from Remark 1.13, in particular the identification of $\Upsilon_{\gamma,\sigma}\tau_c$ with the gradient of the scalar curvature in the geometric case. □

Remark 4.10 In this "geometric" situation, both divergent terms appearing in (1.10) vanish, so that in this case one also obtains finite limits $\bar U^\varrho(\Gamma,\sigma) = \lim_{\varepsilon\to 0} U_\varepsilon^{\text{geo}}(\Gamma,\sigma)$

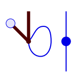
without the need for any renormalisation at all! However, these limits do in general depend on the choice of isometric embedding, since they differ by some vector field of the form $\Upsilon_{\Gamma,\sigma}\tau$ with $\tau \in \mathscr{V}$, and these are sensitive to both the choice of σ_i's and of the mollifier ϱ.

5 General theory of T-algebras

We now introduce an algebraic structure that encodes the space of "formal expressions built from derivatives of σ_i and Γ with indices contracted according to Einstein's summation convention" which include the functions $\Upsilon_{\Gamma,\sigma}\tau$ considered above. It is then natural to look for a structure for a space that exhibits the following features:

- It should be graded by pairs (u,ℓ) of integers denoting the number of "upper" and "lower" indices respectively. Furthermore, on each such space, the symmetric group $\mathrm{Sym}(u,\ell) = \mathrm{Sym}(u)\times\mathrm{Sym}(\ell)$ should act by permuting the corresponding indices. An example of element of degree $(1,2)$ would be $\partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta\partial_\zeta\sigma_2^\eta$. It is of degree $(1,2)$ because there are two free lower indices (γ and ζ) and one free upper index (η).
- It should come with a product $\bullet$ turning it into a (bi)graded algebra (for example $\partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta \bullet \partial_\zeta\sigma_2^\eta = \partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta\partial_\zeta\sigma_2^\eta$), a "partial trace" tr performing a contraction of the last two indices (so for example $\mathrm{tr}\,\partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta\partial_\zeta\sigma_2^\eta = \partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta\partial_\eta\sigma_2^\eta$), as well as a derivation ∂.

The aim of the next subsection is to exhibit a number of intertwining relations that are naturally satisfied by these operations and to provide a justification for the use of diagrammatic notations for general expressions of the type mentioned above. For example, the diagrammatic notation yields

$$\partial_\alpha\Gamma^\alpha_{\beta\gamma}\sigma_1^\beta\partial_\zeta\sigma_2^\eta =$$ 

Here, ◦ denotes σ_1, • denotes σ_2, ⊙ denotes Γ, thick edges | denote those edges that terminate on one of the "original" indices for a given function, and thin edges | denotes those that terminate on one of the indices generated by a derivation. The "half-edges" that are only connected to one vertex denote the free indices, with "upper indices" represented by the outgoing half edges terminating at the bottom, while the "lower indices" are represented by the incoming half edges starting at the top.

5.1 Definition of a T-algebra

For $u \in \mathbf{N}$ we write $[u] := \{i \in \mathbf{N} : 1 \le i \le u\}$ and $\mathrm{Sym}(u)$ for the symmetric group on $[u]$. We also set $\mathrm{Sym}(u,\ell) = \mathrm{Sym}(u) \times \mathrm{Sym}(\ell)$ where $\times$ denotes the usual direct product of groups. Given $u_1, u_2 \in \mathbf{N}$ and $\alpha_i \in \mathrm{Sym}(u_i)$, we define the concatenation $\alpha_1 \cdot \alpha_2 \in \mathrm{Sym}(u_1+u_2)$ of α_1 and α_2 as

$$\alpha_1 \cdot \alpha_2\,(i) := \alpha_1(i)\mathbb{1}_{(i\le u_1)} + (u_1 + \alpha_2(i-u_1))\mathbb{1}_{(i>u_1)}, \qquad i \in [u_1+u_2].$$

This yields natural embeddings

$$\begin{aligned}\mathrm{Sym}(u_1,\ell_1)\times\mathrm{Sym}(u_2,\ell_2)&\to\mathrm{Sym}(u_1+u_2,\ell_1+\ell_2)\\(\alpha_1,\alpha_2)&\mapsto\alpha_1\cdot\alpha_2\;,\end{aligned}$$

but note that $\alpha_1\cdot\alpha_2\neq\alpha_2\cdot\alpha_1$. In this context, it will be convenient to write $S^{u_1,u_2}_{\ell_1,\ell_2}$ for the element of $\mathrm{Sym}(u_1+u_2,\ell_1+\ell_2)$ 'swapping the two factors' so that

$$S^{u_1,u_2}_{\ell_1,\ell_2}(\alpha_1\cdot\alpha_2)=(\alpha_2\cdot\alpha_1)S^{u_1,u_2}_{\ell_1,\ell_2}\;.$$

We omit an index if its value vanishes (this is unambiguous) and we write id^u_ℓ for the identity on $\mathrm{Sym}(u,\ell)$. (So we could actually write S^u_ℓ instead of id^u_ℓ.) First, we formalise the notion of a tensor algebra $\mathcal{V}$ with trace in an abstract way.

Definition 5.1 A tensor algebra with trace $\mathcal{V}$ consists of a (bi)graded vector space $\mathcal{V}=\bigoplus\{\mathcal{V}^u_\ell\,:\,u,\ell\geq 0\}$ together with the following additional data.

- For every $u,\ell\geq 0$, a left action of $\mathrm{Sym}(u,\ell)$ onto $\mathcal{V}^u_\ell$.
- For any $u_i,\ell_i\geq 0$, a product $\bullet\colon\mathcal{V}^{u_1}_{\ell_1}\times\mathcal{V}^{u_2}_{\ell_2}\to\mathcal{V}^{u_1+u_2}_{\ell_1+\ell_2}$ which is associative and satisfies the coherence properties

$$B\bullet A=S^{u_1,u_2}_{\ell_1,\ell_2}(A\bullet B)\;,\quad\alpha_1 A\bullet\alpha_2 B=(\alpha_1\cdot\alpha_2)(A\bullet B)\;,\tag{5.1}$$

  for any $\alpha_i\in\mathrm{Sym}(u_i,\ell_i)$. Furthermore, this product admits a unit $\mathbf{1}\in\mathcal{V}^0_0$.
- For any $u,\ell\geq 0$, a linear map $\mathrm{tr}\colon\mathcal{V}^{u+1}_{\ell+1}\to\mathcal{V}^u_\ell$ such that

$$\alpha\,\mathrm{tr}\,A=\mathrm{tr}\big((\alpha\cdot\mathrm{id}^1_1)A\big),\qquad\forall\,\alpha\in\mathrm{Sym}(u,\ell),\;A\in\mathcal{V}^{u+1}_{\ell+1},\tag{5.2}$$

$$\mathrm{tr}^2 A=\mathrm{tr}^2\left((\mathrm{id}^u_\ell\cdot S^{1,1}_{1,1})A\right)\,,\qquad\forall\,A\in\mathcal{V}^{u+2}_{\ell+2},\tag{5.3}$$

$$\mathrm{tr}(A\bullet B)=A\bullet\mathrm{tr}\,B\;,\qquad\forall\,A\in\mathcal{V},\qquad B\in\mathcal{V}^{u+1}_{\ell+1}\;.\tag{5.4}$$

Remark 5.2 As usual, we don't actually need $\mathcal{V}$ to be a vector space, but could consider modules over a ring instead.

Remark 5.3 We will say that $\tau\in\mathcal{V}$ has degree (u,ℓ) to indicate that $\tau\in\mathcal{V}^u_\ell$. Note that the identity (5.4) only holds if B belongs to some $\mathcal{V}^u_\ell$ with both ℓ and u at least one.

Remark 5.4 An example of a tensor algebra with trace is obtained by choosing any vector space V and setting $\mathcal{V}[V]=\bigoplus_{u,\ell}\mathcal{V}^u_\ell[V]$ where

$$\mathcal{V}^u_\ell[V]=(V^*)^{\otimes\ell}\otimes V^{\otimes u}\;,\qquad u,\ell\geq 0,$$

and V^* is the dual space of V. The group $\mathrm{Sym}(u,\ell)$ acts in the natural way by permuting the factors in the sense that, for $\alpha=(\alpha_u,\alpha_\ell)\in\mathrm{Sym}(u,\ell)$, we set

$$\alpha(v_1^*\otimes\ldots\otimes v_\ell^*\otimes v_1\otimes\ldots\otimes v_u)\tag{5.5}$$

$$= v^*_{\alpha_\ell^{-1}(1)} \otimes \ldots \otimes v^*_{\alpha_\ell^{-1}(\ell)} \otimes v_{\alpha_u^{-1}(1)} \otimes \ldots \otimes v_{\alpha_u^{-1}(u)} \;.$$

The product is the usual tensor product, except that one sets

$$(A_1 \otimes B_1) \bullet (A_2 \otimes B_2) = (A_1 \otimes A_2) \otimes (B_1 \otimes B_2)\;,$$

with $A_i \in (V^*)^{\otimes \ell_i}$ and $B_i \in V^{\otimes u_i}$. The operator tr is defined by

$$\mathrm{tr}\left((f_1 \otimes \ldots \otimes f_{\ell+1}) \otimes (v_1 \otimes \ldots \otimes v_{u+1})\right) = f_{\ell+1}(v_{u+1})(f_1 \otimes \ldots \otimes f_\ell) \otimes (v_1 \otimes \ldots \otimes v_u)\;.$$

It is easy to check that this satisfies all the properties mentioned above.

Definition 5.5 A T-algebra $\mathscr{V}$ is a tensor algebra with trace which furthermore admits a *derivation*, namely a collection of linear maps $\partial \colon \mathscr{V}_\ell^u \to \mathscr{V}_{\ell+1}^u$ with the following coherence properties for $A \in \mathscr{V}_\ell^u$ and $\alpha \in \mathrm{Sym}(u,\ell)$:

$$\partial(\alpha A) = (\mathrm{id}_1^0 \cdot \alpha)\partial A\;, \qquad \partial^2 A = (S_{1,1} \cdot \mathrm{id}_\ell^u)\partial^2 A\;. \tag{5.6}$$

We furthermore assume that ∂ behaves "nicely" with respect to both $\bullet$ and tr in the sense that for $A_i \in \mathscr{V}_{\ell_i}^{u_i}$

$$\partial(A_1 \bullet A_2) = \partial A_1 \bullet A_2 + (S_{\ell_1,1}^{u_1} \cdot \mathrm{id}_{\ell_2}^{u_2})(A_1 \bullet \partial A_2)\;, \tag{5.7}$$

as well as

$$\partial\, \mathrm{tr}\, A = \mathrm{tr}\, \partial A\;, \qquad A \in \mathscr{V}_{\ell+1}^{u+1}. \tag{5.8}$$

Remark 5.6 Any T-algebra $\mathscr{V}$ comes with a natural bilinear "grafting" operation $\curvearrowright$ on $\mathscr{V}_0^1$ given by

$$A \curvearrowright B = \mathrm{tr}(\partial B \bullet A)\;.$$

It is an instructive exercise to verify that this turns $\mathscr{V}_0^1$ into a pre-Lie algebra. For this, one first verifies that

$$A \curvearrowright (B \curvearrowright C) - (A \curvearrowright B) \curvearrowright C = \mathrm{tr}^2(\partial^2 C \bullet A \bullet B) \tag{5.9}$$

and then notices that this is symmetric in A, B thanks to (5.3) and the second identity in (5.6).) In particular, $\mathscr{V}_0^1$ is a Lie algebra with Lie bracket $[A,B] = A \curvearrowright B - B \curvearrowright A$.

Remark 5.7 The second identity in (5.6) encodes the fact that we are considering "flat" spaces at the algebraic level. Indeed, this is the only identity that breaks if we take for $\mathscr{V}$ the tensor bundle over a smooth manifold and for ∂ the covariant derivative. Removing this condition however would break the pre-Lie structure of $\mathscr{V}_0^1$ mentioned above.

Remark 5.8 Combining both identities in (5.6) shows that one also has

$$\partial^m A = (\alpha \cdot \mathrm{id}_\ell^u)\partial^m A\;,$$

for every $m \geq 1$ and every $\alpha \in \mathrm{Sym}(0, m)$. Similarly, (5.2) and (5.3) show that

$$\mathrm{tr}^m A = \mathrm{tr}^m((\mathrm{id}_\ell^u \cdot \alpha)A)\;,$$

for every permutation $\alpha = \delta \times \delta \in \mathrm{Sym}(m, m)$ with $\delta \in \mathrm{Sym}(m)$ and every $A \in \mathcal{V}_{\ell+m}^{u+m}$.

Given two T-algebras $\mathcal{V}$ and $\mathcal{W}$, a linear map $\varphi\colon \mathcal{V} \to \mathcal{W}$ is a morphism of T-algebras if it preserves all of the above structure. Similarly, an *ideal* of T-algebras is a linear subspace that is a two-sided ideal for the multiplication and that is stable under the trace, derivation, and action of the symmetric group.

Given a morphism of T-algebras $\iota\colon \mathcal{V} \to \mathcal{W}$, a linear map $\varphi\colon \mathcal{V} \to \mathcal{W}$ is an infinitesimal morphism with respect to ι if it satisfies the property

$$\varphi(A \bullet B) = \varphi(A) \bullet \iota(B) + \iota(A) \bullet \varphi(B)\;, \tag{5.10}$$

as well as

$$\varphi(\alpha A) = \alpha\varphi(A)\;, \qquad \varphi(\mathrm{tr}\, A) = \mathrm{tr}\,\varphi(A)\;, \qquad \varphi(\partial A) = \partial\varphi(A)\;,$$

for every homogeneous A and every element α of the symmetric group such that these operations make sense. In particular, an infinitesimal morphism is uniquely determined by its action on any set of generators for $\mathcal{V}$. In our case, the morphism ι will usually be given by some natural inclusion $\mathcal{V} \subset \mathcal{W}$, in which case we omit it from the terminology.

Remark 5.9 A typical example of T-algebra relevant for this article is the following. Fix a finite-dimensional vector space V and let $\mathcal{V}[V]$ be the tensor algebra with trace based on V considered in Remark 5.4. We then have a T-algebra $\mathcal{W}[V]$ obtained by taking for $\mathcal{W}_\ell^u$ the space of all smooth functions $V \to \mathcal{V}_\ell^u[V]$. The product, action of the symmetric group, and trace tr are defined pointwise, exactly as in Remark 5.4. The derivation ∂ is then given by the usual Fréchet derivative with the canonical identification

$$L(V, \mathcal{V}_\ell^u) \approx V^* \otimes \mathcal{V}_\ell^u[V] \approx \mathcal{V}_{\ell+1}^u[V]\;.$$

It is a straightforward exercise to verify that it satisfies the required coherence conditions, with the second part of (5.6) a consequence of the fact that the second derivative is symmetric and (5.7) a consequence of the Leibniz rule.

5.2 Characterisation of free T-algebras

We now give another very natural collection of T-algebras. Fix a finite number of 'types' $\mathfrak{X}$ and, for each type $\mathfrak{t} \in \mathfrak{X}$, fix integers $i_\mathfrak{t}$ and $o_\mathfrak{t}$ which we interpret as 'incoming' and 'outgoing' slots respectively. (Call this data a 'bigraded set $\mathfrak{X}$'.) We also set $i^\star_\mathfrak{t} = [i_\mathfrak{t}] \sqcup \{\star\}$. Given such an $\mathfrak{X}$, we consider the following collection of objects.

Definition 5.10 Given a bigraded set $\mathfrak{X}$, an $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)^u_\ell$ of degree (u, ℓ) consists of a finite vertex set V_g, a type map $\mathfrak{t}\colon V_g \to \mathfrak{X}$, and a map $\varphi\colon \mathrm{Out}(g) \to \mathrm{In}(g)$, where

$$\begin{aligned}
\mathrm{Out}(g) &= [\ell] \sqcup \overline{\mathrm{Out}}(g)\;, & \mathrm{In}(g) &= [u] \sqcup \overline{\mathrm{In}}(g)\;,\\
\overline{\mathrm{Out}}(g) &= \bigsqcup_{v\in V_g} \{v\} \times [o_{\mathfrak{t}(v)}]\;, & \overline{\mathrm{In}}(g) &= \bigsqcup_{v\in V_g} \{v\} \times i^\star_{\mathfrak{t}(v)}\;,
\end{aligned}$$

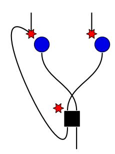
with the property that every element of $[u] \sqcup \bigsqcup_{v\in V_g} \{v\} \times [i_{\mathfrak{t}(v)}]$ has exactly one preimage under φ and

$$\varphi^{-1}([u]) \cap [\ell] = \emptyset\;. \tag{5.11}$$

(Note that targets of type $\star$ can have any number of preimages, including none.) We interpret elements of $\mathrm{Out}(g)$ as 'directed edges' via the maps $\mathrm{dom}\colon \mathrm{Out}(g) \to [\ell]\times V_g$ and $\mathrm{cod}\colon \mathrm{Out}(g) \to [u] \times V_g$ defined by setting $\mathrm{dom}\restriction[\ell] = \mathrm{id}$, $\mathrm{dom}(v, j) = v$, $\mathrm{cod}(e) = \mathrm{cod}^\star(\varphi(e))$, with $\mathrm{cod}^\star\restriction[u] = \mathrm{id}$, $\mathrm{cod}^\star(v, j) = v$. We also define the set of 'internal edges' of g by

$$\mathrm{Intern}(g) = \overline{\mathrm{Out}}(g) \setminus \varphi^{-1}([u]).$$

Remark 5.11 As already mentioned, we will sometimes view an $\mathfrak{X}$-graph as an actual directed graph with vertex set $\hat{V}_g = V_g \sqcup [u] \sqcup [\ell]$ and edge set $\mathrm{Out}(g)$, with dom mapping an edge to its source and cod mapping it to its target. See the figure below for a graphical representation of an $\mathfrak{X}$-graph when $\mathfrak{X}$ is composed of two elements ● $\in \mathfrak{X}^1_0$ and ■ $\in \mathfrak{X}^2_2$.



Condition (5.11) guarantees that every edge in this graph is adjacent to at least one vertex of V_g. Those edges that are adjacent to an element of $[\ell]$ (resp. $[u]$) should be thought of as incoming (resp. outgoing) half-edges with the order of $[\ell]$ (resp. $[u]$) providing a distinguished enumeration for them.

Remark 5.12 Given an $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)^u_\ell$, write d_v with $v \in V_g$ for the number of preimages of $(v, \star)$. Then, the conditions imposed on φ guarantee that $\sum_{v\in V_g} o_v - u = \sum_{v\in V_g}(i_v + d_v) - \ell = |\mathrm{Intern}(g)| \geq 0$.

As usual, we identify any two $\mathfrak{X}$-graphs $g_i = (V_i, \mathfrak{t}_i, \varphi_i)^u_\ell$ if there exists a bijection $\iota\colon V_1 \to V_2$ such that $\mathfrak{t}_1 = \mathfrak{t}_2 \circ \iota$ and such that $\varphi_2 \circ \iota_{\mathrm{Out}} = \iota_{\mathrm{In}} \circ \varphi_1$, where

$$\iota_{\mathrm{Out}}\colon \mathrm{Out}(g_1) \to \mathrm{Out}(g_2)\;, \qquad \iota_{\mathrm{In}}\colon \mathrm{In}(g_1) \to \mathrm{In}(g_2)\;, \tag{5.12}$$

are the natural bijections induced by ι (given by the identity on $[u]$ and $[\ell]$ respectively). We call such a bijection an $\mathfrak{X}$-graph isomorphism.

Definition 5.13 Given any $\mathfrak{X}$-graph g as above, we write $\mathscr{G}_g$ for the group of internal symmetries of g, namely the group of all $\mathfrak{X}$-graph isomorphisms from g to itself.

Definition 5.14 For any $\mathfrak{t} \in \mathfrak{X}$ the *elementary* $\mathfrak{X}$-graph of type $\mathfrak{t}$ is the graph $(\{\bullet\}, \mathfrak{t}, \varphi)_{i_\mathfrak{t}}^{o_\mathfrak{t}}$, where φ is the natural injection $[i_\mathfrak{t}] \sqcup (\{\bullet\} \times [o_\mathfrak{t}]) \to [o_\mathfrak{t}] \sqcup (\{\bullet\} \times i_\mathfrak{t}^\star)$ such that $\varphi^{-1}(\bullet, \star) = \emptyset$.

We then define $\mathcal{V}_\ell^u$ as the vector space generated by all $\mathfrak{X}$-graphs of degree (u, ℓ) and the unit $\mathbf{1}$ as the unique $\mathfrak{X}$-graph with empty vertex set. (This is necessarily of degree $(0, 0)$ by (5.11).) An element $\alpha = (\alpha_u, \alpha_\ell)$ in the symmetric group $\mathrm{Sym}(u, \ell)$, with $\alpha_u \in \mathrm{Sym}(u)$ and $\alpha_\ell \in \mathrm{Sym}(\ell)$, acts naturally on an $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)_\ell^u$ as follows:

$$\alpha g := (V_g, \mathfrak{t}, \alpha\varphi)_\ell^u,$$

where $\alpha\varphi\colon \mathrm{Out}(g) \to \mathrm{In}(g)$ is given by

$$(\alpha\varphi)(r) = \begin{cases} \varphi \circ \alpha_\ell^{-1}(r) & \text{for } r \in [\ell], \\ \alpha_u \circ \varphi(r) & \text{for } \varphi(r) \in [u], \\ \varphi(r) & \text{otherwise.} \end{cases}$$

The product $g_1 \cdot g_2$ of two $\mathfrak{X}$-graphs is given by

$$(V_1, \mathfrak{t}_1, \varphi_1)_{\ell_1}^{u_1} \cdot (V_2, \mathfrak{t}_2, \varphi_2)_{\ell_2}^{u_2} = (V_1 \sqcup V_2, \mathfrak{t}_1 \sqcup \mathfrak{t}_2, \varphi_1 \sqcup \varphi_2)_{\ell_1+\ell_2}^{u_1+u_2} ,$$

where we identify $\mathrm{Out}(g_1) \sqcup \mathrm{Out}(g_2)$ with $\mathrm{Out}(g_1 \cdot g_2)$, and we set $V = V_1 \sqcup V_2$, $\mathfrak{t} = \mathfrak{t}_1 \sqcup \mathfrak{t}_2$, and $\varphi_1 \sqcup \varphi_2 : \mathrm{Out}(g_1) \sqcup \mathrm{Out}(g_2) \to \mathrm{In}(g_1) \sqcup \mathrm{In}(g_2)$ is defined in an obvious way. The trace $\mathrm{tr}(V_g, \mathfrak{t}, \varphi)_{\ell+1}^{u+1}$ is given by the graph $(V_g, \mathfrak{t}, \hat\varphi)_\ell^u$ where $\hat\varphi$ agrees with φ except that, for the (unique by assumption) element $r \in \mathrm{Out}(g)$ such that $\varphi(r) = u + 1$, one sets $\hat\varphi(r) = \varphi(\ell + 1)$. Since $u + 1$ is the only element in the domain of φ that we are excluding from the domain of $\hat\varphi$, the result is again an $\mathfrak{X}$-graph. Both properties (5.3) and (5.4) are immediate.

Finally, the operation ∂ is defined by setting, for $g = (V_g, \mathfrak{t}, \varphi)_\ell^u$,

$$\partial g = \sum_{v \in V_g} \partial_v g \,, \tag{5.13}$$

where $\partial_v g = (V_g, \mathfrak{t}, \partial_v\varphi)_{\ell+1}^u$ with $\partial_v\varphi = \varphi$ outside of $[\ell + 1]$ and

$$\partial_v\varphi(1) = (v, \star) \,, \qquad \partial_v\varphi(j) = \varphi(j-1) \,, \quad j \in \{2, \ldots, \ell+1\} \,.$$

The identity (5.8) and the first property of (5.6) are immediate. The identity in (5.6) follows from the fact that $\partial_u\partial_v\varphi$ is obtained by precomposing $\partial_u\partial_v\varphi$ with the permutation of $[\ell + 2]$ that swaps the first two values. Finally, (5.7) follows from the fact that $\partial_v(\varphi_1 \sqcup \varphi_2)$ is obtained from $\varphi_1 \sqcup \partial_v\varphi_2$ by precomposing it with the map $S_{\ell_1,1}$.

Remark 5.15 Definition 5.14 gives us a natural way of identifying $\mathcal{X}$ with a subset of the T-algebra of $\mathcal{X}$-graphs.

Theorem 5.16 *For any bigraded set $\mathcal{X}$, the T-algebra $T(\mathcal{X})$ of $\mathcal{X}$-graphs is the free T-algebra generated by $\mathcal{X}$. The bigrading is such that $T^u_\ell(\mathcal{X})$ is spanned by the graphs of degree (u, ℓ).*

Remark 5.17 The notion of freeness is the usual one, namely that for any T-algebra $\mathcal{W}$ and any map $\Phi\colon \mathcal{X} \to \mathcal{W}$ with $\Phi(\mathfrak{t}) \in \mathcal{W}^{o_\mathfrak{t}}_{i_\mathfrak{t}}$, there exists a unique extension $\Phi : T(\mathcal{X}) \to \mathcal{W}$ of Φ to a morphism of T-algebras.

In order to prepare for the proof of this result, we first present a few preliminary definitions and results. Given any $m \geq 1$, and any (u, ℓ) with $u \wedge \ell \geq m$, we also have a 'diagonal' representation D of the symmetric group $\mathrm{Sym}(m)$ on $T^u_\ell(\mathcal{X})$ as follows. Given $\sigma \in \mathrm{Sym}(m)$, we write $D_\sigma \in \mathrm{Sym}(u, \ell)$ for the element

$$D_\sigma = (\mathrm{id}_{u-m} \cdot \sigma) \times (\mathrm{id}_{\ell-m} \cdot \sigma) .$$

This then acts on $g \in T^u_\ell(\mathcal{X})$ via the representation of $\mathrm{Sym}(u, \ell)$ described above. We intentionally omit the indices u, ℓ and m from the notations since these are always uniquely determined from the context. With these notations at hand, we have the following crucial preliminary result.

Lemma 5.18 *Every $\mathcal{X}$-graph g can be written in the form*

$$g = \mathrm{tr}^m \, \alpha(\partial^{d_1}\mathfrak{t}_1 \bullet \ldots \bullet \partial^{d_n}\mathfrak{t}_n) , \tag{5.14}$$

with the convention that $g = \mathbf{1}$ if $n = 0$. Furthermore, if one also has

$$g = \mathrm{tr}^m \, \alpha'(\partial^{d'_1}\mathfrak{t}'_1 \bullet \ldots \bullet \partial^{d'_n}\mathfrak{t}'_n) ,$$

then there exist elements $\mu \in \mathrm{Sym}(n)$, $\nu \in \mathrm{Sym}(m)$ and $\sigma_i \in \mathrm{Sym}(d_i)$ for $i = 1, \ldots, n$ with the following properties.

1. *One has $d'_i = d_{\mu(i)}$ and $\mathfrak{t}'_i = \mathfrak{t}_{\mu(i)}$ for every i.*
2. *The element α' is related to α by $\alpha' = D_\nu \cdot \alpha \cdot \hat\sigma \cdot \hat\mu$, where $\hat\mu \in \mathrm{Sym}(u+m, \ell+m)$ is the unique element such that*

$$\hat\mu(V_{\mu(1)} \cdots V_{\mu(n)}) = V_1 \cdots V_n ,$$

for any T-algebra $\mathcal{V}$ and any collection of elements $V_i \in \mathcal{V}$ such that V_i has the same degree as $\partial^{d_i}\mathfrak{t}_i$. Similarly, $\hat\nu \in \mathrm{Sym}(u + m, \ell + m)$ is the unique element such that

$$\hat\nu(\partial^{d_1}W_1 \cdots \partial^{d_n}W_n) = \partial^{d_1}W_1 \cdots \partial^{d_n}W_n ,$$

for any T-algebra $\mathcal{V}$ and any collection of elements $W_i \in \mathcal{V}$ such that W_i has the same degree as $\mathfrak{t}_i$.

Proof. The statement is trivial for $g = \mathbf{1}$, so we assume that $g = (V_g, \mathfrak{t}, \varphi)^u_\ell$, we write $n = |V_g| > 0$, and set $m = |\mathrm{Intern}(g)|$. We then consider an ordering $[n] \ni i \mapsto v_i \in V_g$ of its vertices, as well as an ordering $[m] \ni i \mapsto e_i \in \mathrm{Intern}(g)$ of its internal edges. For every $i \in [n]$, fix furthermore an element $\sigma_i \in \mathrm{Sym}(d_i)$. Let us write $T^{\mathrm{ex}}(\mathfrak{X})$ for the space of all $\mathfrak{X}$-graphs equipped with this additional data and $\pi^{\mathrm{ex}} \colon T^{\mathrm{ex}}(\mathfrak{X}) \to T(\mathfrak{X})$ for the linear map forgetting this data. This map has a natural right inverse $\iota^{\mathrm{ex}} \colon T(\mathfrak{X}) \to T^{\mathrm{ex}}(\mathfrak{X})$ obtained by averaging over all possible assignments of the additional data (v, e, σ).

We claim that $T^{\mathrm{ex}}(\mathfrak{X})$ is the natural space in which a representation of the type (5.14) lives in the sense that this additional data yields a unique representation of g of the type (5.14). Indeed, note first that since we've ordered the vertices, we can naturally set

$$\mathfrak{t}_i = \mathfrak{t}(v_i)\;, \qquad d_i = |\varphi^{-1}(v_i, \star)|\;.$$

We then obtain a natural ordering $\eta_1 \colon \overline{\mathrm{Out}}(g) \to [u+m]$ by choosing it to be the unique bijection such that $\eta_1(v_i, j) < \eta_1(v_k, \ell)$ if and only if $i < k$ or $i = k$ but $j < \ell$.

On the other hand, we obtain a natural map $\eta_2 \colon [\ell + m] \to \overline{\mathrm{In}}(g)$ by setting $\eta_2(j) = \varphi(j)$ for $j \in [\ell]$ and, for $j \in [m]$, we set $\eta_2(\ell + j) = \varphi(\eta_1^{-1}(u + j))$. Using these maps, as well as the map $e \colon [m] \to \mathrm{Intern}(g)$ from our data, we build $\alpha = \alpha_1 \times \alpha_2 \in \mathrm{Sym}(u + m, \ell + m)$ as follows.

For α_1, we set

$$\alpha_1^{-1}(j) = \begin{cases} \eta_1(\varphi^{-1}(j)) & \text{if } j \le u, \\ \eta_1(e_{j-u}) & \text{otherwise.} \end{cases}$$

Regarding α_2, we set it to be the only bijection such that

$$\alpha_2^{-1}(j) \in \begin{cases} \eta_2^{-1}(\varphi(j)) & \text{if } j \le \ell, \\ \eta_2^{-1}(\varphi(e_{j-u})) & \text{otherwise.} \end{cases}$$

and such that furthermore the order of α_2^{-1} on $\eta_2^{-1}(v_i, \star)$ is the same as the order given by σ_i. It is tedious but straightforward to show that this choice of α guarantees that (5.14) holds. Furthermore, any two representations of g built in this way differ from (5.14) by an element of $\mathrm{Sym}(m) \times \mathrm{Sym}(n) \times \big(\bigtimes_{i-1}^{n} \mathrm{Sym}(d_i)\big)$ representing a relabelling of the internal edges, of the vertices, and a change of choice of the σ_i.

The fact that g cannot be represented in any other way (except for taking linear combinations of such representations) immediately follows from the fact that ι^{ex} is injective. □

Proof of Theorem 5.16. Given any $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)^u_\ell$, we write

$$g = \mathrm{tr}^m\, \alpha(\partial^{d_1}\mathfrak{t}_1 \bullet \ldots \bullet \partial^{d_n}\mathfrak{t}_n)\;.$$

We then set

$$\Phi(g) = \mathrm{tr}^m\, \alpha(\partial^{d_1}\Phi(\mathfrak{t}_1) \bullet \ldots \bullet \partial^{d_n}\Phi(\mathfrak{t}_n))\;.$$

It follows from Lemma 5.18, combined with Remark 5.8 that Φ is well-defined, so that it suffices to check that it is a morphism of T-algebras. The fact that $\Phi(\mathrm{tr}\, g) = \mathrm{tr}\, \Phi(g)$ if $\ell \wedge u \geq 1$ and that $\Phi(\sigma g) = \sigma\Phi(g)$ for $\sigma \in \mathrm{Sym}(u,\ell)$ follows immediately from the well-posedness of Φ.

Regarding $\partial\Phi(g)$, note that as a consequence of the three identities postulated in Definition 5.5, given any $\alpha \in \mathrm{Sym}(u+m,\ell+m)$ we can find elements $\alpha_i \in \mathrm{Sym}(u+m,\ell+m+1)$ such that the identity

$$\partial\, \mathrm{tr}^m \alpha(V_1 \bullet \ldots \bullet V_n) = \sum_{i=1}^n \mathrm{tr}^m \alpha_i(V_1 \bullet \ldots \bullet \partial V_i \bullet \ldots \bullet V_n)\;,$$

holds true in any T-algebra and for any elements V_i such that the degree of V_i equals that of $\partial^{d_i} \mathfrak{t}_i$. The identity $\partial\Phi(g) = \Phi(\partial g)$ then follows at once. Regarding the product, we use the fact that for any T-algebra and any two elements A and B of degrees at least (ℓ,ℓ) and (m,m) respectively, we can find an element α of the symmetric group such that $\mathrm{tr}^\ell A \bullet \mathrm{tr}^m B = \mathrm{tr}^{\ell+m} \alpha(A \bullet B)$. Indeed, making repeated use of (5.2) and (5.4), we find that there are elements α_1, α_2 and α such that

$$\begin{aligned}\mathrm{tr}^\ell A \bullet \mathrm{tr}^m B &= \alpha_1(\mathrm{tr}^m B \bullet \mathrm{tr}^\ell A) = \alpha_1\, \mathrm{tr}^\ell(\mathrm{tr}^m B \bullet A) = \mathrm{tr}^\ell \alpha_2(A \bullet \mathrm{tr}^m B)\\ &= \mathrm{tr}^\ell \alpha_2\, \mathrm{tr}^m(A \bullet B) = \mathrm{tr}^{\ell+m} \alpha(A \bullet B)\;.\end{aligned}$$

The required claim then follows at once. □

5.3 An injectivity result

Definition 5.19 We say that an $\mathfrak{X}$-graph is *anchored* if every connected component of the associated directed graph defined in Remark 5.11 contains either an element of $[u]$ or an element of $[\ell]$. In other words, $g \in T^u_\ell(\mathfrak{X})$ is anchored if, whenever $g = \tilde g \bullet \tau$ with $\tilde g \in T^u_\ell(\mathfrak{X})$ and $\tau \in T^0_0(\mathfrak{X})$, one must have $\tau = \mathbf{1}$.

Recall the definition of the tensor algebra with trace $\mathscr{V}[V]$ in Remark 5.4. and of the T-algebra $\mathscr{W}[V]$ in Remark 5.9, for a given vector space V.

Definition 5.20 Given a T-algebra $\mathscr{T}$, a subspace $\mathscr{T}_0 \subset \mathscr{T}$ is said to be realisable if one can find a finite-dimensional (real) vector space V and a morphism $\Phi\colon \mathscr{T} \to \mathscr{W}[V]$ such that $\mathscr{T}_0 \ni \tau \mapsto \Phi(\tau)(0) \in \mathscr{V}[V]$ is injective.

Remark 5.21 Not every T-algebra is realisable: consider the case of $\mathscr{T}$ with one generator X quotiented by the ideal generated by the relation $X \bullet X = 0$. Since there exists no non-vanishing real-valued function with vanishing square, the only possible morphism $\Phi\colon \mathscr{T} \to \mathscr{W}[V]$ is to map X to 0, so that the one-dimensional subspace of $\mathscr{T}$ generated by X is not realisable.

Theorem 5.22 *The subspace generated by any finite collection of connected anchored $\mathfrak{X}$-graphs is realisable for the free T-algebra generated by some bigraded set $\mathfrak{X}$.*

Proof of Theorem 5.22. Let $\mathfrak{X}$ be a finite bigraded set and let $\mathfrak{S}$ be any finite collection of connected anchored $\mathfrak{X}$-graphs. We will use the notation $\hat{\pi}$ for the quotient map by action of the symmetric group, namely given $g, g' \in T_\ell^u(\mathfrak{X})$, we write $\hat{\pi}g = \hat{\pi}g'$ if and only if there exists $\alpha \in \mathrm{Sym}(u, \ell)$ such that $g' = \alpha g$. We assume that, for any $g \neq g'$ in $\mathfrak{S}$, one has $\hat{\pi}g \neq \hat{\pi}g'$, but we will then show that there exists a finite-dimensional vector space V and a map $\Phi\colon \mathfrak{X} \to \mathscr{W}[V]$ such that, denoting by $\hat{\Phi}\colon T(\mathfrak{X}) \to \mathscr{W}[V]$ the unique morphism of T-algebras extending Φ, the map $\langle\hat{\mathfrak{S}}\rangle \ni \tau \mapsto \hat{\Phi}(\tau)(0) \in \mathscr{V}[V]$ is injective, where $\hat{\mathfrak{S}}$ consists of all $\mathfrak{X}$-graphs g such that there exists $g' \in \mathfrak{S}$ with $\hat{\pi}g = \hat{\pi}g'$. We fix once and for all an ordering $\mathfrak{S} = \{g_1, \ldots, g_N\}$ and we set $G = (V_G, \mathfrak{t}_G, \varphi_G) = g_1 \bullet \ldots \bullet g_N$. Note that with this representation, every $\mathfrak{X}$-graph $g \in \mathfrak{S}$ is naturally identified with a subgraph of G.

Given an $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)_\ell^u$, we define a map $\mathrm{Type}_g\colon \mathrm{Out}(g) \sqcup \mathrm{In}(g) \to (\mathfrak{X} \times \mathbf{N} \times \mathbf{N}) \sqcup \{\top, \bot\}$ as follows. For $e = (v, j) \in \overline{\mathrm{Out}}(g)$ with $|\varphi^{-1}(v, \star)| = k$, we set $\mathrm{Type}_g(e) = (\mathfrak{t}(v), k, j)$, which we also write as $(\partial^k \mathfrak{t}(v), j)$. If $e = (v, j) \in \overline{\mathrm{In}}(g)$ with $|\varphi^{-1}(v, \star)| = k$, we similarly write $\mathrm{Type}_g(v, j) = (\partial^k \mathfrak{t}(v), j)$ with the convention that we set the last component to 0 if $j = \star$. Finally, we set $\mathrm{Type}_g(e) = \top$ for $e \in [\ell] \subset \mathrm{Out}(g)$ and $\mathrm{Type}_g(e) = \bot$ for $e \in [u] \subset \mathrm{In}(g)$.

Given an arbitrary $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)_\ell^u$, an *$\mathfrak{X}$-graph morphism* from g to G is a map

$$\psi\colon \mathrm{Out}(g) \to \mathrm{Out}(G)\;, \tag{5.15}$$

with the following properties. (Here, we say that two edges $e, e' \in \mathrm{Out}(g)$ 'touch' each other if $(\mathrm{dom}\, e \cup \mathrm{cod}\, e) \cap (\mathrm{dom}\, e' \cup \mathrm{cod}\, e') \neq \emptyset$.)

1. If $e, e' \in \mathrm{Out}(g)$ touch each other, then $\psi(e)$ and $\psi(e')$ also touch each other. Furthermore, they touch 'in the same way' in the sense that for any $f_i \in \{\mathrm{dom}, \mathrm{cod}\}$, $f_1(e) = f_2(e')$ implies that $f_1(\psi(e)) = f_2(\psi(e'))$.
2. For every $v \in V_g$, ψ is injective on $\varphi^{-1}(v, \star)$.
3. On $\overline{\mathrm{Out}}(g)$, we have the identity

$$\mathrm{Type}_g = \mathrm{Type}_G \circ \psi\;, \tag{5.16}$$

while on $\mathrm{Intern}(g) \sqcup [\ell]$ we have the identity

$$\mathrm{Type}_g \circ \varphi = \mathrm{Type}_G \circ \varphi_G \circ \psi\;. \tag{5.17}$$

We denote by $\mathscr{E}(g, G)$ the set of all $\mathfrak{X}$-graph morphisms from g to G. We say that such a morphism is *anchored* if (5.16) and (5.17) extend to all of $\mathrm{Out}(g)$ and the map ψ is injective on $\mathrm{Out}(g) \setminus \mathrm{Intern}(g)$.

We now choose $V = \mathbf{R}^{\mathrm{Out}(G)}$ and write $\{E_u : u \in \mathrm{Out}(G)\}$ for its canonical basis and $\{E_u^\star : u \in \mathrm{Out}(G)\}$ for the dual basis. For any $\mathfrak{X}$-graph $g \in T_\ell^u(\mathfrak{X})$ and any $\psi \in \mathscr{E}(g, G)$, we then define an element $U(g, \psi) \in \mathscr{V}_\ell^u[V] = (V^*)^{\otimes \ell} \otimes V^{\otimes u}$ by setting

$$U(g, \psi) = \bigotimes_{i \in [\ell]} E_{\psi(i)}^\star \otimes \bigotimes_{j \in [u]} E_{\psi(\varphi^{-1}(j))}\;.$$

For any $\mathfrak{t} \in \mathfrak{X}$ of degree (u, ℓ), we choose a function $\Phi_\mathfrak{t} \colon V \to (V^*)^{\otimes \ell} \otimes V^{\otimes u}$ such that

$$D^k \Phi_\mathfrak{t}(0) = \sum_{\psi \in \mathscr{E}(\partial^k \mathfrak{t}, G)} U(\partial^k \mathfrak{t}, \psi) , \tag{5.18}$$

for k such that there exists $g = (V_g, \mathfrak{t}, \varphi) \in \mathfrak{S}$ which has a vertex $v \in V_g$ of type $\mathfrak{t}$ such that $|\varphi^{-1}(v, \star)| \geq k$. Here $\partial^k \mathfrak{t}$ denotes the elementary graph as in Definition 5.14.

The reason why it is possible to construct such a function Φ is that the right-hand side of (5.18) is necessarily symmetric in its first k factors. Indeed, for any $\sigma \in \mathrm{Sym}(k)$, we can define $P_\sigma \colon \mathrm{Out}(\partial^k \mathfrak{t}) \to \mathrm{Out}(\partial^k \mathfrak{t})$ to be equal to σ on $[k]$ (viewed as the initial segment of $[k + i_\mathfrak{t}] \subset \mathrm{Out}(\partial^k \mathfrak{t})$) and the identity otherwise. We similarly have an action R of $\mathrm{Sym}(k)$ onto $\mathscr{V}^{o_\mathfrak{t}}_{k+i_\mathfrak{t}}[V] = (V^*)^{\otimes k} \otimes (V^*)^{\otimes i_\mathfrak{t}} \otimes V^{\otimes o_\mathfrak{t}}$ by permuting the first k factors. This is such that

$$U(\partial^k \mathfrak{t}, \psi \circ P_\sigma) = R_\sigma U(\partial^k \mathfrak{t}, \psi) ,$$

so that, since composition with P_σ is a bijection on $\mathscr{E}(\partial^k \mathfrak{t}, G)$ by the injectivity constraint implied by properties 2 and 3 on $\mathfrak{X}$-graph morphisms, we do indeed have

$$R_\sigma D^k \Phi_\mathfrak{t}(0) = \sum_{\psi \in \mathscr{E}(\partial^k \mathfrak{t}, G)} U(\partial^k \mathfrak{t}, \psi \circ P_\sigma) = \sum_{\psi \in \mathscr{E}(\partial^k \mathfrak{t}, G)} U(\partial^k \mathfrak{t}, \psi) = D^k \Phi_\mathfrak{t}(0) . \tag{5.19}$$

Let now g be an arbitrary $\mathfrak{X}$-graph. We claim now that

$$\hat{\Phi}(g)(0) = \sum_{\psi \in \mathscr{E}(g, G)} U(g, \psi) . \tag{5.20}$$

By definition, this identity holds on the generators and their derivatives. Furthermore, the right hand side is equivariant under the action of $\mathrm{Sym}(u, \ell)$ in the following way. Given $\alpha = (\alpha_u, \alpha_\ell) \in \mathrm{Sym}(u, \ell)$, $g \in T^u_\ell(\mathfrak{X})$, and $\psi \in \mathscr{E}(g, G)$, we define $\alpha\psi \in \mathscr{E}(\alpha g, G)$ by setting $\alpha\psi \restriction [\ell] = \psi \circ \alpha_\ell^{-1}$ and $\alpha\psi = \psi$ otherwise. It then follows from (5.5) that

$$U(\alpha g, \alpha\psi) = \bigotimes_{i \in [\ell]} E^\star_{\psi(\alpha_\ell^{-1}(i))} \otimes \bigotimes_{j \in [u]} E_{\psi(\varphi^{-1}(\alpha_u^{-1}(j)))} = \alpha U(g, \psi) ,$$

as requested. By Lemma 5.18, it therefore remains to show that (5.20) is stable under multiplication and trace.

For $g = g_1 \bullet g_2$ we have indeed

$$\sum_{\psi_1 \in \mathscr{E}(g_1, G)} U(g_1, \psi_1) \bullet \sum_{\psi_2 \in \mathscr{E}(g_2, G)} U(g_2, \psi_2) = \sum_{\psi \in \mathscr{E}(g, G)} U(g, \psi) ,$$

since an $\mathfrak{X}$-graph morphism ψ of g into G defines by restriction morphisms ψ_1 and ψ_2 of g_1 and g_2, and vice-versa. Furthermore, the three properties we require ψ to satisfy are verified if and only if they are verified for their restrictions to g_1 and g_2.

This is because they only depend on the local connectivity structure of the $\mathfrak{X}$-graph g and this does not 'see' distinct connected components.

We now turn to the trace operation. Note that one has

$$\mathrm{Out}(\mathrm{tr}\, g) = \mathrm{Out}(g) \setminus \{\ell\}\;.$$

Furthermore, $\mathfrak{X}$-graph morphisms $\hat\psi \in \mathscr{E}(\mathrm{tr}\, g, G)$ are in natural one-to-one correspondence with $\mathfrak{X}$-graph morphism $\psi \in \mathscr{E}(g, G)$ that coincide with $\hat\psi$ on $\mathrm{Out}(\mathrm{tr}\, g)$ and such that $\psi(\ell) = \psi(\varphi^{-1}(u))$. Indeed, since one has natural identifications $\overline{\mathrm{Out}}(g) = \overline{\mathrm{Out}}(\mathrm{tr}\, g)$ and $\mathrm{Intern}(g) \sqcup [\ell] = \mathrm{Intern}(\mathrm{tr}\, g) \sqcup [\ell-1]$, and since ψ coincides with $\hat\psi$ modulo these identifications, properties 2 and 3 above hold for ψ if and only if they hold for $\hat\psi$. The same is true for property 1, which can be verified by a simple case analysis.

For $\ell, u \geq 1$, we therefore conclude that one has for $g \in T_\ell^u(\mathfrak{X})$

$$\begin{aligned}
&\mathrm{tr} \sum_{\psi\in\mathscr{E}(g,G)} \bigotimes_{j=1}^{u} E^\star_{\psi(j)} \otimes \bigotimes_{k=1}^{\ell} E_{\psi(\varphi^{-1}(k))} = \\
&= \sum_{\psi\in\mathscr{E}(g,G)} \mathbf{1}_{\psi(u)=\psi(\varphi^{-1}(\ell))} \bigotimes_{j=1}^{u-1} E^\star_{\psi(j)} \otimes \bigotimes_{k=1}^{\ell-1} E_{\psi(\varphi^{-1}(k))} \\
&= \sum_{\psi\in\mathscr{E}(g,G)} \mathbf{1}_{\psi(u)=\psi(\varphi^{-1}(\ell))} U(\mathrm{tr}\, g, \psi) = \sum_{\hat\psi\in\mathscr{E}(\mathrm{tr}\, g,G)} U(\mathrm{tr}\, g, \hat\psi)\;,
\end{aligned}$$

thus concluding the proof of (5.20).

It remains to show that the map $\hat{\mathfrak{S}} \ni g \mapsto \hat\Phi(g)(0) \in \mathscr{V}[V]$ yields linearly independent elements. For this, it suffices to construct for every $g \in \hat{\mathfrak{S}}$ a linear functional $f_g : \mathscr{V}[V] \to \mathbf{R}$ such that

$$f_g(\hat\Phi(\tau)(0)) = \langle g, \tau\rangle, \qquad \forall\, \tau \in \langle \mathfrak{S}\rangle, \tag{5.21}$$

where the scalar product makes distinct elements of $\hat{\mathfrak{S}}$ orthogonal. The natural normalisation for this scalar product is the one such that $\langle g, g\rangle = |\mathscr{G}_g|$, similarly to what we have already observed for $\mathcal{S}$. The construction of f_g follows quite easily from the following result, the proof of which will be given below.

Lemma 5.23 *Assume that we have two connected and anchored $\mathfrak{X}$-graphs $g_i = (V_i, \mathfrak{t}_i, \varphi_i)_\ell^u$ with anchored $\mathfrak{X}$-graph morphisms ψ_i of g_i into G such that $\psi_1 = \psi_2$ on $[\ell]$ and $\psi_1 \circ \varphi_1^{-1} = \psi_2 \circ \varphi_2^{-1}$ on $[u]$. Then, there exists an $\mathfrak{X}$-graph isomorphism $\iota\colon V_1 \to V_2$ (in the sense of (5.12)) such that furthermore $\psi_1 = \psi_2 \circ \iota_{\mathrm{Out}}$.*

Assume that this holds for the moment. Recall also that, for every $g \in \mathfrak{S}$, there is a canonical (anchored) $\mathfrak{X}$-graph morphism $\psi_g\colon \mathrm{Out}(g) \to \mathrm{Out}(G)$. Similarly to above, we then set

$$f_g = U^\star(g, \psi_g) = \bigotimes_{i\in[\ell]} E_{\psi_g(i)} \otimes \bigotimes_{j\in[u]} E^\star_{\psi_g(\varphi^{-1}(j))}\;.$$

Note that $f_g \in (\mathscr{V}_\ell^u[V])^*$ so that this is indeed a linear functional on $\mathscr{V}[V]$. (We extend it so that it vanishes on components of degree other than (u,ℓ).) For $\alpha g \in \hat{\mathfrak{G}}$ with $g \in \mathfrak{G}$, we similarly set

$$f_{\alpha g} = U^\star(\alpha g, \alpha\psi_g)\;.$$

Combining these definitions with (5.20), we see that $f_g(\hat\Phi(\tau)(0))$ vanishes unless the degree of τ equals that of g and then, it precisely equals the number of $\mathscr{X}$-graph morphisms ψ from τ into G such that $\psi = \psi_g$ on $[\ell]$ and $\psi \circ \varphi^{-1} = \psi_g \circ \psi_g^{-1}$ on $[u]$. By Lemma 5.23, this vanishes unless $\tau = g$, and in that case it equals $|\mathscr{G}_\tau| = \langle g, \tau\rangle$ as required. □

Proof of Lemma 5.23. In a first step, we note that one must have range $\psi_1 =$ range ψ_2. Indeed, let $\mathscr{A} \subset \mathrm{Out}(G)$ be the subset constructed recursively as follows. We set $\mathscr{A}_0 = \psi_i([\ell]) \cup (\psi_i \circ \varphi_i^{-1})([u])$ which belongs to the range of both ψ_i by definition. Then, given $\mathscr{A}_k$, we let $\mathscr{A}_{k+1}$ be the set of all edges $e \in \mathrm{Out}(G)$ touching some edge $e' \in \mathscr{A}_k$. Assuming that $\mathscr{A}_k \subset$ range ψ_i, the definition of an $\mathscr{X}$-graph morphism then implies that the same is true for $\mathscr{A}_{k+1}$, so that $\mathscr{A} = \bigcup_k \mathscr{A}_k \subset$ range ψ_i. On the other hand, the assumption that the g_i are connected guarantees that range $\psi_i = \mathscr{A}$.

In a second step, we show that the ψ_i are actually injective. Assume by contradiction that ψ_1 (say) is not injective, so that there exist edges $e \neq e'$ such that $\psi_1(e) = \psi_1(e') = \hat e \in \mathscr{A}$. Since $\mathscr{A}$ is connected, we can find a path $(\hat e_k)_{k=0}^n$ of edges in $\mathscr{A}$ such that $\hat e_0 = \hat e$, $\hat e_n \in \mathrm{Out}(G) \setminus \mathrm{In}(G)$ and such that, for every $k < n$, $\hat e_k$ and $\hat e_{k+1}$ touch each other in the same sense as in the previous paragraph. The definition of an $\mathscr{X}$-graph morphism (in particular the 'local' injectivity of ψ_1 on the set of edges adjacent to any given vertex implied by properties 2 and 3) then implies that there are unique paths e_k and e_k' with $\psi_1(e_k) = \psi_1(e_k') = \hat e_k$ and such that e_k and e_{k+1} touch each other, and similarly for e_k'. Furthermore, these paths must stay disjoint since, if $e_k \neq e_k'$, then e_k and e_k' cannot touch each other (again by local injectivity), so that one must also have $e_{k+1} \neq e_{k+1}'$. It follows that $e_n \neq e_n'$, but these must both belong to $\mathrm{Out}(g_1) \setminus \mathrm{In}(g_1)$ by the definition of the $\mathscr{X}$-graph morphism being anchored, which contradicts the fact that ψ_1 is injective there.

Since both ψ_i are bijections between $\mathrm{Out}(g_i)$ and $\mathscr{A}$, this yields a unique bijection $\hat\iota\colon \mathrm{Out}(g_1) \to \mathrm{Out}(g_2)$ such that $\psi_1 = \psi_2 \circ \hat\iota$. One can verify that one actually has $\hat\iota \in \mathscr{E}(g_1, g_2)$. The required map ι is now obtained by mapping $v \in V_1$ to the unique $\iota(v) \in V_2$ such that dom $e = v$ implies dom $\hat\iota(e) = \iota(v)$ and similarly for cod e. One can verify that this is indeed a graph isomorphism, thus concluding the proof. □

Remark 5.24 The assumptions that elements of $\mathfrak{G}$ are connected can easily be dropped. Indeed, the only place where this was used is in order to obtain the injectivity of the ψ_i in the proof of Lemma 5.23. If connectedness is dropped, then we can only conclude that the ψ_i are injective on each connected component of g_i. This can be avoided by choosing $V = \mathbf{R}^{\mathrm{Out}(G)} \otimes \mathbf{R}^X$ for X a finite set of 'colours' and replacing the set $\mathscr{E}(g,G)$ by a set $\mathscr{E}_X(g,G)$ of 'coloured' $\mathscr{X}$-graph morphisms which furthermore assign a colour to each connected component of g. The conclusion of Lemma 5.23 then still holds, provided that we furthermore enforce that ψ_1 assigns a distinct colour to each connected component of g_1 and the colour assignment of ψ_2 agrees with that of ψ_1 on the inputs and outputs.

Corollary 5.25 *If $\mathcal{T}$ is a T-algebra such that there exists an injective morphism ι from $\mathcal{T}$ into a free T-algebra, then every subspace of the form $\langle \iota^{-1}(\mathfrak{S})\rangle$ with $\mathfrak{S}$ a finite collection of connected anchored graphs is realisable.*

Proof. Just consider the morphism $\Phi \circ \iota$ with ι the injective morphism from the assumption. □

In some situations, generators of our T-algebras have additional symmetries reflecting symmetries of the actual functions they represent, as is the case for example for the generator representing Γ, which is symmetric in its two lower indices. The following injectivity result which covers this type of situation. For this, given a bigraded set $\mathfrak{X}$ and the corresponding T-algebra $T(\mathfrak{X})$ as above, we write $T(\mathfrak{X}) = \bigoplus_{k\geq 1} T_k(\mathfrak{X})$, where $T_k(\mathfrak{X})$ is generated by the $\mathfrak{X}$-graphs with exactly k nodes.

Corollary 5.26 *Let $\mathfrak{X}$ be as above and, for every $A \in \mathfrak{X}$, fix a subgroup $S_A \subset$ $\mathrm{Sym}(u,\ell)$, where (u,ℓ) is the degree of A. Then, the T-algebra $\hat{T}(\mathfrak{X})$ obtained by quotienting $T(\mathfrak{X})$ by the ideal of T-algebras generated by all elements of the form $(A - \alpha A)$ with $\alpha \in S_A$ admits an injective morphism into $T(\mathfrak{X})$.*

Proof. Write $\pi\colon T(\mathfrak{X}) \to \hat{T}(\mathfrak{X})$ for the canonical projection. Let now $\iota\colon \hat{T}(\mathfrak{X}) \to T(\mathfrak{X})$ be the morphism such that

$$\iota(\pi(A)) = \frac{1}{|S_A|}\sum_{\alpha\in S_A} \alpha A\;.$$

We claim that this uniquely defines ι and that it is a right inverse for π. Indeed, one has $\iota(\alpha\pi(A)) = \iota(\pi(A))$ for every $\alpha \in S_A$, so it is well-defined on the quotient space $\hat{T}(\mathfrak{X})$, and one has $\pi(\iota(\pi(A))) = \frac{1}{|S_A|}\sum_{\alpha\in S_A}\pi(\alpha A) = \pi(A)$ as required. Being a right inverse, ι is injective, so the claim follows. □

It is sometimes useful to restrict ourselves a priori to maps Φ that are close to a given Φ_0, see below the discussion after (6.12). To formulate the corresponding statement, we note first that given a finite set $\mathfrak{S}$ of $\mathfrak{X}$-graphs and a map $\Phi\colon \mathfrak{X} \to \mathcal{W}[V]$ as above, there exists some $N \geq 0$ such that $(\hat{\Phi}\tau)(0)$ is a multilinear map of $\{D^k\Phi_{\mathfrak{t}}(0) \,:\, \mathfrak{t}\in\mathfrak{X},\ k\leq N\}$. For the purpose of this section, we can therefore identify maps Φ for which these quantities coincide, so that Φ is viewed as an element of the finite-dimensional vector space $\mathcal{Y}_N$ consisting of maps that are polynomial of degree at most N. With this notation at hand, we have

Corollary 5.27 *Let $\mathfrak{X}$ be as above and let $\mathfrak{S}$ be a finite collection of anchored $\mathfrak{X}$-graphs. Then, for any neighbourhood $\mathcal{U}$ of $\mathcal{Y}_N$, the map $\langle\mathfrak{S}\rangle \to (\mathcal{U} \to \mathcal{V}[V])$*

$$\tau \mapsto (\Psi \mapsto (\hat{\Psi}\tau)(0))$$

is injective.

Proof. Fix $\Psi_0 \in \mathcal{U}$ and write Ψ_t for the map such that $\Psi_t(A) = \Psi_0(A) + t\Phi(A)$ for $A \in \mathfrak{X}$, with Φ as in Theorem 5.22. Note now that if τ is 'homogeneous' of degree k (in the sense that it is an $\mathfrak{X}$-graph g with $|V_g| = k$), then one has $\partial_t^k \hat{\Psi}_t(\tau)|_{t=0} = k!\hat{\Phi}(\tau)$. The claim then follows by decomposing any arbitrary τ into its homogeneous components. □

Remark 5.28 As in Corollary 5.25, the statement remains true for any T-algebra $\mathcal{T}$ and subspace $V \subset \mathcal{T}$ provided that there exists an injective morphism from $\mathcal{T}$ into a free T-algebra which maps V onto a subspace of the space generated by anchored graphs.

5.4 Hilbert space structure

We have already seen that the free T-algebra $T(\mathfrak{X})$ comes with a natural scalar product such that $\langle g, g\rangle = |\mathcal{G}_g|$ and such that $\langle g, h\rangle = 0$ if the $\mathfrak{X}$-graphs g and h aren't isomorphic. It is then natural to ask what are the adjoints of the four defining operations of a T-algebra. It is immediate that, for $\alpha \in \mathrm{Sym}(u, \ell)$, one has $\alpha^* = \alpha^{-1}$. Regarding the product, we define its adjoint $\Delta\colon T(\mathfrak{X}) \to T(\mathfrak{X}) \otimes T(\mathfrak{X})$ to be the linear map such that

$$\langle f \bullet g, h\rangle = \langle f \otimes g, \Delta h\rangle \;. \tag{5.22}$$

To characterise Δ, we say that an $\mathfrak{X}$-graph h is irreducible if whenever $h = f \bullet g$ one must have either $f = \mathbf{1}$ or $g = \mathbf{1}$.

Irreducible graphs of degree $(0, 0)$ play a special role since they commute with every element of $T(\mathfrak{X})$. For every $\mathfrak{X}$-graph h, we can then find a unique (up to permutation) collection of distinct irreducible graphs $(g_i)_{i=1}^n$ of degree $(0, 0)$ as well as a collection of exponents k_i and irreducible graphs $(f_j)_{j=1}^m$ such that

$$h = f_1 \bullet \ldots \bullet f_m \bullet g^k \;, \qquad g^k = \prod_{i=1}^n g_i^{k_i} \;. \tag{5.23}$$

With this notation at hand, we claim that one has

$$\Delta h = \sum_{p \le m} \sum_{\ell \le k} \binom{k}{\ell} (f_1 \bullet \ldots \bullet f_p) \bullet g^\ell \otimes (f_{p+1} \bullet \ldots \bullet f_m) \bullet g^{k-\ell} \;, \tag{5.24}$$

again with the conventions that empty products are interpreted as $\mathbf{1}$, that $\ell \le k$ means $\ell_i \le k_i$ for $0 \le i \le n$ and $\binom{k}{\ell} = \prod_{i=1}^n \binom{k_i}{\ell_i}$.

From the uniqueness of the decomposition (5.23), it follows that Δh as characterised by (5.22) must be of the type (5.24), with the binomial coefficient replaced by some coefficient depending on p, ℓ and h.

At this stage, we note that one has

$$\langle h,h\rangle = |\mathscr{G}_h| = k!\Big(\prod_{i=1}^m |f_i|^2\Big)\Big(\prod_{j=1}^n |g_j|^2\Big)\;,$$

since any isomorphism for h is obtained by composing isomorphisms for the irreducible factors with a permutation of the identical factors g_i. (The f_i cannot be permuted even if some of them are identical as a consequence of the fact that they are anchored.) The claim then follows at once by verifying that the binomial coefficient is the only choice that guarantees that (5.23) holds if we take for f and g the left (resp. right) factor in one of the summands of (5.24).

Regarding the trace operation, we claim that for $g = (V_g, \mathfrak{t}, \varphi)^u_\ell$ one has the identity

$$\mathrm{tr}^* g = \sum_{e\in \mathrm{Out}(g)} \mathrm{Cut}_e(g)\;, \tag{5.25}$$

where $\mathrm{Cut}_e(g) = (V_g, \mathfrak{t}, \hat\varphi)^{u+1}_{\ell+1}$ is obtained by setting $\hat\varphi(e) = u+1$, $\hat\varphi(\ell+1) = \varphi(e)$, and $\hat\varphi = \varphi$ otherwise. Again, the fact that it is of this type is obvious from the definition of tr (in particular in view of the representation given by Lemma 5.18), so we only need to verify that, for every $e \in \mathrm{Out}(g)$, one has $|g|^2 = N_e|\,\mathrm{Cut}_e\, g|^2$, where N_e denotes the number of edges $\hat e$ such that $\mathrm{Cut}_{\hat e}(g) = \mathrm{Cut}_e(g)$.

On the other hand, viewing elements of $\mathscr{G}_g$ as bijections on $\mathrm{Out}(g)$ via (5.12), it is straightforward to verify that $\mathscr{G}_{\mathrm{Cut}_e(g)}$ is naturally identified with the subgroup of $\mathscr{G}_g$ that fixes e, while N_e is nothing but the size of the orbit of e under the action of $\mathscr{G}_g$. The claim (5.25) now follows from the orbit-stabiliser theorem.

Finally, for g as above, we have $\partial^* g = 0$ unless $\ell > 0$ and $\varphi(1) = (v,\star)$ for some $v \in V_g$. In that case, we claim that $\partial^* g = \hat g = (V_g, \mathfrak{t}, \hat\varphi)^u_{\ell-1}$ where $\hat\varphi(j) = \varphi(j+1)$ for $j \in [\ell]$ and $\hat\varphi = \varphi$ otherwise. Again, it is clear that $\partial^* g$ must be some multiple of $\hat g$. Furthermore, by definition of the adjoint, we would like to have

$$\langle \hat g, \partial^* g\rangle = \langle \partial \hat g, g\rangle = \hat N_g \langle g,g\rangle\;,$$

where $\hat N_g$ is the number of vertices $\hat v$ of $\hat g$ such that $\partial_{\hat v}\hat g = g = \partial_v \hat g$, with $\partial_{\hat v}\hat g$ defined as in (5.13) and v the specific vertex of V_g singled out above. If we now interpret $\mathscr{G}_{\hat g}$ as a group of bijections of V_g, we note again that $\hat N_g$ is nothing but the orbit of v under $\mathscr{G}_{\hat g}$, while $\mathscr{G}_g$ is the stabiliser of v, so that the claim follows again from the orbit-stabiliser theorem.

6 Application to SPDEs

We now return to the reason for introducing all of this algebraic machinery, namely the proof of Proposition 3.15. We first use this to provide a characterisation of the space $\mathcal{S}_{\mathrm{geo}}$.

6.1 Characterisation of geometric counterterms

We consider from now on smooth one-parameter families $(\psi_t)_{t\geq 0}$ of diffeomorphisms of $\mathbf{R}^d$ such that

$$\psi_0 = \mathrm{id} : \mathbf{R}^d \to \mathbf{R}^d, \qquad \partial_t \psi_t|_{t=0} = h : \mathbf{R}^d \to \mathbf{R}^d. \tag{6.1}$$

Definition 6.1 We write $\bar{\mathcal{S}}_\circ$ for the free T-algebra generated by $\{\circ_i, \odot : i = 1,\ldots,m\}$ with $\circ_i$ of degree $(1,0)$ and $\odot$ of degree $(1,2)$, quotiented by the ideal of T-algebras generated by $(\odot - S^1_{1,1}\odot)$. We define $\bar{\mathcal{S}}_{\circ,\blacktriangle}$ in the same way as $\bar{\mathcal{S}}_\circ$ by adding an additional element $\blacktriangle$ of degree $(1,0)$ to the set of generators.

We will henceforth view the space $\mathcal{S}_\circ$ defined on page 17 (and therefore also the reduced space $\mathcal{S} \subset \mathcal{S}_\circ$) as a subspace of $\bar{\mathcal{S}}_\circ$ in the obvious way, namely each tree $\tau \in \mathfrak{S}_\circ$ is viewed as an $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi)$ where the vertex set V_g is given by the inner vertices of τ, the type of a vertex is given by σ_i if it is incident to such a noise, while it is given by $\odot$ otherwise. The map φ is determined by the edges of τ, with thin edges representing connections to the corresponding instance of $(v,\star)$, while thick edges represent connections to the 'native' incoming slots of $\odot$.

In this way of representing elements of $\mathcal{S}_\circ$, we have for example

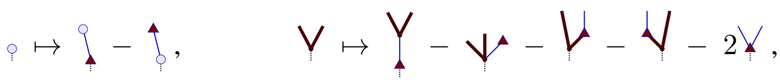
 $\approx \mathrm{tr}^3(\partial\odot \bullet \circ_k \bullet \circ_j \bullet \mathrm{tr}(\partial\circ_i \bullet \circ_\ell))$.

(With this way of writing we avoid having to use the action of the symmetric group.) For any fixed choice of Γ and σ, we then write $\Upsilon_{\Gamma,\sigma} : \bar{\mathcal{S}}_\circ \to \mathscr{W}[\mathbf{R}^d]$ for the (unique) morphism of T-algebras mapping $\circ_i$ to σ_i and $\odot$ to 2Γ, where the target space $\mathscr{W}[\mathbf{R}^d]$ is the T-algebra constructed in Remark 5.9. It is straightforward to convince oneself that this is consistent with our previous notation in the sense that $\Upsilon_{\Gamma,\sigma}$ is an extension of the valuation given in Section 2.3 to all of $\bar{\mathcal{S}}_\circ$. The reason why $\odot$ is mapped to 2Γ and not just to Γ is that in $\bar{\mathcal{S}}_\circ$ every instance of $\odot$ always has two incoming thick edges, so that one performs two derivatives in the q variables for the corresponding vertex in the formula (2.6).

Given furthermore a function $h : \mathbf{R}^d \to \mathbf{R}^d$, we extend $\Upsilon_{\Gamma,\sigma}$ to a morphism of T-algebras $\Upsilon^h_{\Gamma,\sigma} : \bar{\mathcal{S}}_{\circ,\blacktriangle} \to \mathscr{W}[\mathbf{R}^d]$ by sending $\blacktriangle$ to h. The map $\varphi_{\mathrm{geo}} : \bar{\mathcal{S}}_\circ \to \bar{\mathcal{S}}_{\circ,\blacktriangle}$ is the unique infinitesimal morphism of T-algebras with respect to the canonical injection $\iota : \bar{\mathcal{S}}_\circ \to \bar{\mathcal{S}}_{\circ,\blacktriangle}$ such that

$$\begin{aligned}\varphi_{\mathrm{geo}}(\circ_i) &= [\circ_i, \blacktriangle] = \circ_i \curvearrowright \blacktriangle - \blacktriangle \curvearrowright \circ_i\,,\\ \varphi_{\mathrm{geo}}(\odot) &= [\odot, \blacktriangle] - \mathrm{tr}(S^{1,1}_3(\partial\blacktriangle \bullet \odot)) - S^1_{1,1}\,\mathrm{tr}(S^{1,1}_3(\partial\blacktriangle \bullet \odot)) - 2\partial^2\blacktriangle\,,\end{aligned} \tag{6.2}$$

see (5.10). Pictorially, φ_{geo} acts by setting

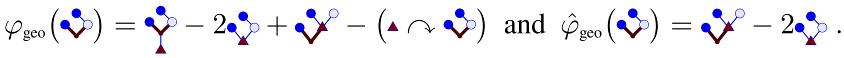


which furthermore has a very natural interpretation in terms of how vector fields and Christoffel symbols transform under infinitesimal changes of coordinates, see

(6.5) below. Here, we used the graphical convention that edges that end in one of the two 'original' slots of $\odot$ are drawn as thick lines, while edges that end in one of the additional slots created by the operator ∂ are drawn as thin lines. We also set

$$\hat{\varphi}_{\rm geo}(\tau) = \varphi_{\rm geo}(\tau) - [\tau, \blacktriangle]\;,$$

so that one has for example



(6.3)

With all of these notations at hand, we are ready to give the following characterisation of the space $\mathcal{S}_{\rm geo}$.

Proposition 6.2 *One has* $\mathcal{S}_{\rm geo} = \mathcal{S} \cap \ker \hat{\varphi}_{\rm geo}$.

Proof. Recall first the action (1.6) of the group of diffeomorphisms on connections and tensors and Definition 3.2 of $\mathcal{S}_{\rm geo}$, where the action on $\Upsilon_{\Gamma,\sigma}$ is that on vector fields as given by (1.6b). It follows that if $(\psi_t)_{t\geq 0}$ is as in (6.1) then, for any $\tau \in \mathcal{S}$

$$\begin{aligned}\partial_t(\psi_t \bullet \Upsilon_{\Gamma,\sigma}\tau)|_{t=0} &= \partial_t((\partial_\alpha\psi_t \cdot \Upsilon^\alpha_{\Gamma,\sigma}\tau)\circ\psi_t^{-1})|_{t=0} \\ &= \partial_\alpha h \cdot \Upsilon^\alpha_{\Gamma,\sigma}\tau - \partial_\alpha\Upsilon_{\Gamma,\sigma}\tau \cdot h^\alpha = \Upsilon^h_{\Gamma,\sigma}[\tau,\blacktriangle]\;.\end{aligned}\tag{6.4}$$

On the other hand, it is easy to see from the definitions that the derivative of a one-parameter family $t\mapsto\varphi_t$ of morphisms of T-algebras is an infinitesimal morphism of T-algebras at φ_t and is therefore defined by its actions on any set of generators. For the family $t\mapsto\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}$, we conclude from (1.6) (see also [Yan57, Eq. 2.16] for example) that

$$\begin{aligned}\partial_t\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}(\circ_i)|_{t=0} &= [\sigma_i, h]\;, \qquad (6.5)\\ \partial_t\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}(\odot)^\alpha_{\beta\gamma}|_{t=0} &= \partial_\eta h^\alpha\Gamma^\eta_{\beta\gamma} - \Gamma^\alpha_{\eta\beta}\partial_\gamma h^\eta - \Gamma^\alpha_{\eta\gamma}\partial_\beta h^\eta - h^\eta\partial_\eta\Gamma^\alpha_{\beta\gamma} - \partial^2_{\beta\gamma}h^\alpha\;,\end{aligned}$$

so that, for every $\tau\in\bar{\mathcal{S}}_\circ$,

$$\partial_t\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}(\tau)|_{t=0} = \Upsilon^h_{\Gamma,\sigma}\varphi_{\rm geo}(\tau)\;.\tag{6.6}$$

This is because both are infinitesimal morphisms at $\Upsilon_{\Gamma,\sigma}$ and they agree on the generators by (6.5).

Comparing (6.6) and (6.4), we see that if $\tau\in\mathcal{S}_{\rm geo}$, then one necessarily has

$$\Upsilon^h_{\Gamma,\sigma}\varphi_{\rm geo}(\tau) = \Upsilon^h_{\Gamma,\sigma}[\tau,\blacktriangle]\;,$$

for every choice of Γ, σ and h, and we conclude that $\tau\in\ker\hat{\varphi}_{\rm geo}$ by Corollary 5.26.

Conversely, let $\tau\in\mathcal{S}\cap\ker\hat{\varphi}_{\rm geo}$ and fix any diffeomorphism ψ homotopic to the identity. We then write $\{\psi_t\}_{t\in[0,1]}$ for any smooth homotopy connecting the identity to ψ and set $\hat{\psi}_t = \psi\circ\psi_t^{-1}$. Let furthermore $h_t = \partial_t\psi_t\circ\psi_t^{-1}$ and $\hat{h}_t = \partial_t\hat{\psi}_t\circ\hat{\psi}_t^{-1}$

and note that as a consequence of the identity $\partial_t(\hat\psi_t \circ \psi_t) = 0$, one obtains the relation

$$\hat h_t = -\hat\psi_t \bullet h_t \;. \tag{6.7}$$

Performing the same calculation as (6.6) and (6.4) for the expression $t \mapsto \Phi_t \stackrel{\text{def}}{=} \hat\psi_t \bullet (\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}\tau)$, but exploiting the fact that for any vector space V on which the group of diffeomorphisms acts with some action $\bullet$ and any $g \in V$, one has

$$\partial_t(\psi_t \bullet g) = \partial_s(\psi_{t+s}\circ\psi_t^{-1})|_{s=0} \bullet (\psi_t \bullet g) = h_t \bullet (\psi_t \bullet g)\;,$$

we obtain

$$\partial_t \Phi_t = \hat\psi_t \bullet (\Upsilon^{h_t}_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}\varphi_{\mathrm{geo}}(\tau)) + [\hat h_t, \Phi_t]\;.$$

Using (6.7), we obtain

$$[\hat h_t, \Phi_t] = -\hat\psi_t \bullet [h_t, \Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}\tau] = -\hat\psi_t \bullet (\Upsilon^{h_t}_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}[\blacktriangle,\tau])\;,$$

so that $\partial_t\Phi_t = \hat\psi_t \bullet (\Upsilon^{h_t}_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma}\hat\varphi_{\mathrm{geo}}(\tau)) = 0$, since $\tau \in \ker\hat\varphi_{\mathrm{geo}}$. It follows that $\Phi_0 = \Phi_1$, which is precisely the desired identity. $\square$

6.2 Characterisation of Itô counterterms

Definition 6.3 Let $\bar{\mathcal{S}}_{\Box}$ (resp. $\bar{\mathcal{S}}_{\Box,\blacktriangle}$) be the free T-algebra generated by $\{\Box, \odot\}$ (resp. $\{\Box, \odot, \blacktriangle\}$), with $\odot$ (and $\blacktriangle$) as before and $\Box$ of degree $(2,0)$, quotiented by the ideal generated by $(\odot - S^1_{1,1}\odot)$. We then write $\varphi_{\text{Itô}}\colon \bar{\mathcal{S}}_{\Box} \to \bar{\mathcal{S}}_{\circ}$ (resp. $\varphi^{\blacktriangle}_{\text{Itô}}\colon \bar{\mathcal{S}}_{\Box,\blacktriangle} \to \bar{\mathcal{S}}_{\circ,\blacktriangle}$) for the morphism mapping $\odot$ to $\odot$ (and $\blacktriangle$ to $\blacktriangle$) and $\Box$ to $\sum_i(\circ_i \bullet \circ_i)$.

We also define $\mathcal{S}_{\blacktriangle} = \varphi_{\mathrm{geo}}(\mathcal{S})$ and $\mathcal{S}^{\blacktriangle}_{\text{Itô}} \subset \mathcal{S}_{\blacktriangle}$ which consists of the elements $\tau \in \mathcal{S}_{\blacktriangle}$ such that $\Upsilon^h_{\Gamma,\sigma}\tau = \Upsilon^h_{\Gamma,\bar\sigma}\tau$ holds for any choice of Γ, σ, $\bar\sigma$ and h such that $\sigma\sigma^\top = \bar\sigma\bar\sigma^\top$.

Proposition 6.4 *One has $\mathcal{S}_{\text{Itô}} = \mathcal{S} \cap \operatorname{range}\varphi_{\text{Itô}}$ and $\mathcal{S}^{\blacktriangle}_{\text{Itô}} = \mathcal{S}_{\blacktriangle} \cap \operatorname{range}\varphi^{\blacktriangle}_{\text{Itô}}$.*

Proof. We focus on proving $\mathcal{S}_{\text{Itô}} = \mathcal{S} \cap \operatorname{range}\varphi_{\text{Itô}}$ and the proof $\mathcal{S}^{\blacktriangle}_{\text{Itô}} = \mathcal{S}_{\blacktriangle} \cap \operatorname{range}\varphi^{\blacktriangle}_{\text{Itô}}$ works exactly the same. The inclusion $\mathcal{S} \cap \operatorname{range}\varphi_{\text{Itô}} \subset \mathcal{S}_{\text{Itô}}$ follows from the fact that $\Upsilon_{\Gamma,\sigma}\circ\varphi_{\text{Itô}}\colon \bar{\mathcal{S}}_{\Box} \to \mathscr{W}[\mathbf{R}^d]$ is a morphism of T-algebras which maps $\odot$ to 2Γ and $\Box$ to g, so that it only depends on Γ and g, and not on the specific choice of σ.

For the converse direction, we want to show the following: for any $\tau \in \mathcal{S} \setminus \operatorname{range}\varphi_{\text{Itô}}$, we can find $\Gamma, \sigma, \bar\sigma$ such that $\sigma\sigma^\top = \bar\sigma\bar\sigma^\top$ but $\Upsilon_{\Gamma,\sigma}\tau \neq \Upsilon_{\Gamma,\bar\sigma}\tau$, implying that $\tau \notin \mathcal{S}_{\text{Itô}}$.

We first show that it is possible to choose a finite-dimensional vector space V such that the conclusion of Theorem 5.22 and its corollaries holds for $\mathfrak{S} = \hat{\mathfrak{S}}$ (with $\hat{\mathfrak{S}}$ the basis of $\mathcal{S}$ as in Section 2.4), but with V *independent* of the number of noise components m. This will then allow us to choose m sufficiently large in order to obtain enough degrees of freedom in our construction. For this, we introduce a T-algebra $\bar{\mathcal{S}}_{\bullet} = \hat{\mathcal{S}}_{\bullet}/\ker\psi$, where $\hat{\mathcal{S}}_{\bullet}$ is the free T-algebra with generators $\odot$ of degree $(1,2)$ and $\{\boxed{k|\ell}\}_{k,\ell\geq 0}$ of degree $(2,k+\ell)$, and $\psi\colon \hat{\mathcal{S}}_\bullet \to \bar{\mathcal{S}}_\circ$ is the unique morphism such that

$$\odot \mapsto \odot\;, \qquad \boxed{k|\ell} \mapsto \sum_i (\partial^k \circ_i \bullet \partial^\ell \circ_i)\;. \tag{6.8}$$

It also follows from our construction that $\mathcal{S} \subset \operatorname{range}\psi$, so that since $\psi : \bar{\mathcal{S}}_\bullet \to \bar{\mathcal{S}}_\circ$ is injective we can (and will) interpret $\mathcal{S}$ as a subspace of $\bar{\mathcal{S}}_\bullet$. With this identification, $\varphi_{\text{Itô}}$ can be viewed as the map $\varphi_{\text{Itô}}\colon \bar{\mathcal{S}}_{\square} \to \bar{\mathcal{S}}_\bullet$ sending $\odot$ to $\odot$ and $\square$ to $\boxed{0|0}$. In particular, range $\varphi_{\text{Itô}}$ is nothing but the intersection of $\mathcal{S}$ with the sub-T-algebra of $\bar{\mathcal{S}}_\bullet$ generated by $\odot$ and $\boxed{0|0}$.

Remark 6.5 Note that $\mathcal{S}$ is strictly smaller than the subspace of $\bar{\mathcal{S}}_\bullet$ generated by $\mathfrak{X}$-graphs of degree $(1,0)$ with either one or two generators of type $\boxed{k|\ell}$. This is because the latter does for example contain the $\mathfrak{X}$-graph obtained by taking the generator $\boxed{1|0}$ and connecting its first output to its input. The map $\Upsilon^\alpha_{\Gamma,\sigma} \circ \psi$ then maps this to the expression $\partial_\eta \sigma_i^\eta \sigma_i^\alpha$ which is 'disconnected' and therefore not generated by any of the graphs on Page 16. If on the other hand one connects the *second* output of $\boxed{1|0}$ to its input, then its image under ψ is given by .

We next identify the kernel of the morphism ψ. Denote by $I \subset \hat{\mathcal{S}}_\bullet$ the smallest ideal of T-algebras enforcing the identities

$$S^{1,1}_{k,\ell}\boxed{k|\ell} = \boxed{\ell|k}\;, \qquad (\alpha\cdot\beta)\boxed{k|\ell} = \boxed{k|\ell}\;, \qquad \forall\,\alpha \in \operatorname{Sym}(1,k),\ \beta \in \operatorname{Sym}(1,\ell)\;, \tag{6.9}$$

as well as $S^1_{1,1}\odot = \odot$ and

$$\partial\boxed{k|\ell} = \boxed{k+1|\ell} + (S^1_{k,1}\cdot \mathrm{id}^1_\ell)\boxed{k|\ell+1}\;. \tag{6.10}$$

It is easy to check that $I \subset \ker\psi$ and we will see below that one actually has equality, but we will not make use of this fact just yet. We then have the following crucial result, the proof of which is postponed to the end of the current proof. This result guarantees in particular that Corollary 5.25 applies to $\hat{\mathcal{S}}_\bullet/I$.

Lemma 6.6 *The canonical projection* $\pi\colon \hat{\mathcal{S}}_\bullet \to \hat{\mathcal{S}}_\bullet/I$ *admits a right-inverse* $\iota\colon \hat{\mathcal{S}}_\bullet/I \to \hat{\mathcal{S}}_\bullet$ *which is a morphism of* T*-algebras and satisfies* $(\iota\circ\pi)(\boxed{0|0}) = \boxed{0|0}$.

Let now V be a finite-dimensional vector space such that the conclusion of Corollary 5.27 holds for $\mathfrak{S} \subset \hat{\mathcal{S}}_\bullet$ given by all graphs of degree $(1,0)$ with at most two vertices of type $\boxed{k|\ell}$, and such that $k+\ell \leq 3$ for these vertices. Note also that by Lemma 6.6, this conclusion also holds for the canonical projection of $\mathfrak{S}$ onto $\hat{\mathcal{S}}_\bullet/I$. Write $S_{\boxed{k|\ell}}$ for the subgroup of $\operatorname{Sym}(2,k+\ell)$ given by all $(\gamma_1,\gamma_2) \in \operatorname{Sym}(2)\times\operatorname{Sym}(k+\ell)$ such that

$$\gamma_1 = \mathrm{id} \in \operatorname{Sym}(2), \qquad \gamma_2 = (\alpha\cdot\beta) \qquad \text{for some } \alpha \in \operatorname{Sym}(k),\ \beta \in \operatorname{Sym}(\ell).$$

As a group, $S_{\boxed{k|\ell}}$ is isomorphic to $\mathrm{Sym}(k,\ell)$. Our definition of I guarantees that one has $\boxed{k|\ell} = \alpha\boxed{k|\ell}$ in $\hat{\mathfrak{S}}_\bullet/I$.

We then write $W_k = V \otimes (V^*)^{\otimes_s k}$ with $\otimes_s$ denoting the symmetric tensor product. It follows from the definition of the group $S_{\boxed{k|\ell}}$ that for any morphism $\Phi\colon \hat{\mathfrak{S}}_\bullet/I \to \mathscr{W}[V]$, one can view $\Phi(\boxed{k|\ell})(0)$ as an element of $W_k \otimes W_\ell$. This is because it belongs to

$$(V^{\otimes 2} \otimes (V^*)^{\otimes(k+\ell)})/S_{\boxed{k|\ell}} \approx V^{\otimes 2} \otimes (V^*)^{\otimes_s k} \otimes (V^*)^{\otimes_s \ell} \approx W_k \otimes W_\ell\;,$$

where the last isometry is obtained by exchanging the two middle factors. In other words, since all these spaces are finite-dimensional, we can view $\Phi(\boxed{k|\ell})(0)$ as a bilinear form on $W_k^* \times W_\ell^*$. Since the ideal I furthermore enforces the first identity in (6.9), these forms satisfy the additional symmetry relation

$$(\Phi(\boxed{k|\ell})(0))(u,v) = (\Phi(\boxed{\ell|k})(0))(v,u)\;.$$

We now show that, for any fixed $n \geq 0$, there exists $m \geq 1$ and, for each valuation Φ in a suitable open set of $\mathscr{Y}_N$ for some large enough but fixed N (recall the definition of $\mathscr{Y}_N$ just before Corollary 5.27), a choice of σ_i and Γ such that

$$(\Upsilon_{\Gamma,\sigma} \circ \psi\tau)(0) = (\Phi\tau)(0)\;, \tag{6.11}$$

for all $\tau \in \mathfrak{S}$. Regarding Γ, it suffices to choose it in such a way that $D^k\Gamma(0) = (\Phi\partial^k \odot)(0)$, so we only need to explain how to choose m and the σ_i.

For this, we set $W = \bigoplus_{k\leq 3} W_k$ and we make a choice of scalar product for each of the W_k (and therefore also for W). Set then $m = \dim W$ and choose any orthonormal basis in W so that $W \approx \mathbf{R}^m$ via that choice of basis. With this choice, we have a natural identification of each $D^k\sigma$ with a smooth function taking values in $L(W, W_k)$, the space of linear maps from W to W_k. We furthermore write $\sigma = \bigoplus_{i\leq 3}\sigma_i$ with each of the σ_i viewed as a function with values in $L(W_i, W_0)$. Similarly, via the choice of scalar product on the W_k, we can view $\Phi(\boxed{k|\ell})(0)$ as an element of $L(W_\ell, W_k)$. With these identifications in place, it then suffices to choose the σ_i in such a way that $(D^k\sigma_i)(0) = 0$ for $i > k$, $\sigma_0(0) = \sqrt{\Phi(\boxed{0|0})(0)}$, and then recursively for $k > \ell$ by

$$\begin{aligned}
(D^k\sigma_\ell)(0) &= \Big(\Phi(\boxed{k|\ell})(0) - \sum_{m<\ell} (D^k\sigma_m(0))(D^\ell\sigma_m(0))^*\Big)(D^\ell\sigma_\ell(0))^{-1}\;,\\
(D^k\sigma_k)(0) &= \Big(\Phi(\boxed{k|k})(0) - \sum_{m<k} (D^k\sigma_m(0))(D^k\sigma_m(0))^*\Big)^{1/2}\;.
\end{aligned}$$

This is of course only possible if $\Phi(\boxed{0|0})(0)$ is strictly positive definite and the same is true for each of the expressions in the large parenthesis of the expression for $D^k\sigma_k$. However, the set of valuations Φ for which these quantities are all strictly positive definite does form an open set $\mathscr{U}_+$ of $\mathscr{Y}_N$.

At this stage, we have all the ingredients in place to show that I is actually equal to $\ker\psi$ rather than being just contained in it. Assume by contradiction that there exists $\tau \in \ker\psi \setminus I$. Setting $\bar\pi = \iota\circ\pi$ with ι and π as in Lemma 6.6, we set $\bar\tau = \bar\pi\tau$, which also belongs to $\ker\psi \setminus I$ since it differs from τ by an element of I. Since $\bar\tau \neq 0$, it follows from Corollary 5.27 that we can find a valuation $\Phi \in \mathcal{U}_+$ such that $\Phi\bar\tau \neq 0$ and such that there exists a valuation $\Upsilon_{\Gamma,\sigma}$ of $\bar{\mathcal{S}}_\circ$ such that (6.11) holds with τ replaced by $\bar\tau$. Since $\bar\tau \in \ker\psi$, the left hand side of this identity must vanish, thus creating the required contradiction.

In order to complete the proof of Proposition 6.4, it remains to show that, given any valuation $\Phi \in \mathcal{U}_+$ as above, there exists an open neighbourhood $\mathcal{U} \subset \mathcal{U}_+$ containing Φ such that, for all $\bar\Phi \in \mathcal{U}$ with $(\bar\Phi\bar\tau)(0) = (\Phi\bar\tau)(0)$ for $\bar\tau \in \operatorname{range}\varphi_{\text{Itô}}$, one can find σ and $\bar\sigma$ such that $\sigma\sigma^\top = \bar\sigma\bar\sigma^\top$ and

$$(\Upsilon_{\Gamma,\sigma}\psi\tau)(0) = (\Phi\tau)(0)\;, \qquad (\Upsilon_{\Gamma,\bar\sigma}\psi\tau)(0) = (\bar\Phi\tau)(0)\;, \tag{6.12}$$

for all $\tau \in \mathcal{S}$. Let us first show that this does indeed imply the non-trivial inclusion $\mathcal{S}_{\text{Itô}} \subset \mathcal{S}\cap\operatorname{range}\varphi_{\text{Itô}}$. Given any $\tau \in \mathcal{S}\setminus\operatorname{range}\varphi_{\text{Itô}}$, Corollary 5.27 guarantees that we can find a valuation $\bar\Phi \in \mathcal{U}$ such that $(\bar\Phi\tau)(0) \neq (\Phi\tau)(0)$, but such that nevertheless $(\bar\Phi\bar\tau)(0) = (\Phi\bar\tau)(0)$ for all $\bar\tau \in \operatorname{range}\varphi_{\text{Itô}}$. It then immediately follows from (6.12) that one must have $\tau \notin \mathcal{S}_{\text{Itô}}$, which does show that $\tau \in \mathcal{S}_{\text{Itô}} \Rightarrow \tau \in \operatorname{range}\varphi_{\text{Itô}}$ as claimed.

Since all of the maps appearing in (6.12) are morphisms, we only need to guarantee that (6.12) holds on sufficiently many generators to generate all of $\mathcal{S}$, namely on $\odot$ and on $\boxed{k|\ell}$ with $k+\ell\le 3$. It furthermore automatically holds on $\odot$, and $\boxed{k|\ell}$ is related to $\boxed{\ell|k}$ by (6.9), so it suffices to show that it holds on the set $\mathcal{T} = \{\boxed{k|\ell} \,:\, k+\ell\le 3 \text{ and } k\le\ell\}$. We furthermore consider maps $\bar\sigma_t$ given by

$$\bar\sigma_t(x) = \sigma(x)\exp(tA(x))\;, \tag{6.13}$$

where A is a smooth function from V to $L(W,W)$ which is such that $A(0)=0$ and $A(x)$ is antisymmetric for every $x\in V$. We will use the notation $A_{k,\ell}(x)$ for the map $\Pi_k A(x)\Pi_\ell^*\colon W_\ell \to W_k$, where $\Pi_k\colon W\to W_k$ is the corresponding orthogonal projection. Antisymmetry then guarantees that (3.2) holds for any such choice of A, while it imposes the relation

$$A_{k,\ell}^* = -A_{\ell,k}\;. \tag{6.14}$$

Since $\varphi_{\text{Itô}}$ maps $\square$ to $\boxed{0|0}$, its range contains $\partial^m\boxed{0|0}$ for every m. It therefore suffices to show that by varying A and t we can generate valuations that assign any possible value to $(\Upsilon_{\Gamma,\bar\sigma_t}\psi\tau)(0)$ for all τ belonging to a collection $\bar{\mathcal{T}} \subset \mathcal{T}$ with the property that $\bar{\mathcal{T}} \cup \{\partial^m\boxed{0|0} \,:\, m\le 3\}$ generates all of $\mathcal{S}$. Since, by (6.10), one has $\partial^m\boxed{0|0} = \boxed{m|0} + \boxed{0|m} + (\ldots)$ where $(\ldots)$ is obtained from $\boxed{k|\ell}$ with $k+\ell = m$ and $k,\ell>0$, we can choose

$$\bar{\mathcal{T}} = \{\boxed{k|\ell} \,:\, k+\ell\le 3 \text{ and } 0<\ell\le k\} \cup \{\boxed{m|0} - \boxed{0|m} \,:\, 1\le m\le 3\}\;.$$

We will choose A in such a way that $D^kA_{m,\ell} = 0$ unless $m=0$ (or $\ell=0$ by (6.14)) and $k\ge\ell$.

With this particular choice and with $\bar\sigma$ as in (6.13), one then has for $0 < k \le \ell$ the identity

$$\partial_t(\Upsilon_{\Gamma,\bar\sigma_t}\psi\boxed{k|\ell})(0)|_{t=0} = (\sigma_0 D^k A_{0,\ell} D^\ell \sigma_\ell^*)(0) + (\ldots)\;,$$

where all the terms appearing in $(\ldots)$ involve $D^{\bar k} A_{0,\bar\ell}$ with either $\bar k < k$ or $\bar\ell < \ell$. Similarly, one has

$$\partial_t(\Upsilon_{\Gamma,\bar\sigma_t}\psi(\boxed{m|0} - \boxed{0|m}))(0)|_{t=0} = (\sigma_0 D^m (A_{0,0} - A_{0,0}^*)\sigma_0^*)(0) + (\ldots)\;,$$

where $(\ldots)$ only involves terms with a factor $D^k A_{0,0}$ with $k < m$. Since, by the first part of our construction, we can choose σ in such a way that the matrices $D^\ell\sigma_\ell(0)$ are invertible for all choices of valuation Φ in a sufficiently small open set $\mathcal{U}$ and since $\Phi(\boxed{m|0} - \boxed{0|m})$ is necessarily antisymmetric for *any* choice of valuation Φ, this shows that, by choosing A in an appropriate way, we can indeed generate all possible values for $(\Phi\tau)(0)$ with $\tau \in \bar{\mathcal{T}}$ by varying A, as claimed. □

Proof of Lemma 6.6. We look for a morphism $\bar\pi\colon \hat{\mathcal{S}}_\bullet \to \hat{\mathcal{S}}_\bullet$ such that $\bar\pi\odot = \frac{1}{2}(\odot + S^1_{1,1}\odot)$ and such that the following properties hold:

- One has $I \subset \ker\bar\pi$.
- For every k and ℓ, one has $\bar\pi\boxed{k|\ell} = \boxed{k|\ell} + R_{k,\ell}$ where $R_{k,\ell} \in I$ which, when combined with the previous property, guarantees that $\bar\pi$ is idempotent.

If we can build such a map $\bar\pi$, then the desired right inverse ι is the unique morphism such that $\bar\pi = \iota\circ\pi$. Indeed, the existence of such a ι is guaranteed by the fact that $I \subset \ker\bar\pi$, while the second property guarantees that $\pi\circ\iota = \mathrm{id}$.

Before we turn to the construction of $\bar\pi$, we define the ideal I_0 in a similar way to I, but without imposing (6.10), and we consider the canonical projection $\pi_0\colon \hat{\mathcal{S}}_\bullet \to \hat{\mathcal{S}}_\bullet/I_0$. The equivalence class $[\boxed{k|\ell}]$ then consists of all elements that are of the form

$$(\alpha\cdot\beta)\boxed{k|\ell}\;,\qquad\text{or}\qquad (\alpha\cdot\beta)S^{1,1}_{\ell,k}\boxed{\ell|k}\;,$$

with $\alpha \in \mathrm{Sym}(1,k)$ and $\beta \in \mathrm{Sym}(1,\ell)$. Furthermore, we have a natural action of the group

$$\hat S_{\boxed{k|\ell}} \stackrel{\text{def}}{=} \mathbf{Z}_2 \times \mathrm{Sym}(1,k)\times\mathrm{Sym}(1,\ell)$$

onto $[\boxed{k|\ell}]$, so that π_0 admits a section ι_0 defined analogously to the construction in the proof of Corollary 5.26.

We now denote by $\hat{\mathcal{S}}_\bullet^{(1)}$ the subspace of $\hat{\mathcal{S}}_\bullet$ generated by $\mathfrak{X}$-graphs with exactly one vertex, and that vertex is not of type $\odot$. We also write $\mathcal{P}^\partial$ (the reason for this choice of notation will become apparent soon) for the image of $\hat{\mathcal{S}}_\bullet^{(1)}$ under π_0. Since $\bar\pi$ is determined by its values on the generators, it then suffices to find a map $\bar\pi_0\colon \mathcal{P}^\partial \to \mathcal{P}^\partial$ which preserves degrees, is equivariant under the action of the symmetric group and the derivation operation, and which is such that the two properties above hold. The desired map $\bar\pi$ is then given by $\bar\pi = \iota_0\circ\bar\pi_0\circ\pi_0$. Note that we do not need to consider the original product anymore. On the other hand, $\mathscr{P}^\partial$ can be identified in a canonical way with the free non-commutative unital algebra generated by three symbols X, $\bar{X}$, and ∂ under the correspondence

$$\partial^m \boxed{k\,|\,\ell} \sim \partial^m X^k \bar{X}^\ell \;. \tag{6.15}$$

We write $\mathscr{P}^\partial_n \subset \mathscr{P}^\partial$ for the subspace of homogeneous polynomials of degree n. Elements $\alpha \in \mathrm{Sym}(2,n) \sim \mathbf{Z}_2 \times \mathrm{Sym}(n)$ act on $\mathscr{P}^\partial_n$ by having $\mathrm{Sym}(n)$ permuting the factors of each monomial and the generator of $\mathbf{Z}_2$ swapping X and $\bar{X}$ (we will also write this as $P \mapsto \bar{P}$). For example, one has

$$(S_{2,3} \cdot \mathrm{id}_2^2)\partial\boxed{2\,|\,4} \sim X\bar{X}^2\partial X\bar{X}^2 \;, \qquad (S_{2,3}^{1,1} \cdot \mathrm{id}_2^0)\partial\boxed{2\,|\,4} \sim \bar{X}X^2\partial\bar{X}X^2 \;.$$

The image under π_0 of the projection onto $\hat{\mathcal{S}}^{(1)}_{\bullet}$ of the ideal (of T-algebras) I is then equal to the ideal (of algebras!) $\hat{I} \subset \mathscr{P}^\partial$ generated by $\partial - X - \bar{X}$.

We claim that one possible choice for $\bar{\pi}_0$ is given by the unique morphism of algebras such that

$$\bar{\pi}_0\partial = \partial \;, \quad \bar{\pi}_0 X = \frac{1}{2}(X - \bar{X} + \partial) \;, \quad \bar{\pi}_0\bar{X} = \frac{1}{2}(\bar{X} - X + \partial) \;. \tag{6.16}$$

This choice clearly satisfies the identity $\overline{\bar{\pi}_0 P} = \bar{\pi}_0\bar{P}$ so that $\bar{\pi}_0$ is indeed equivariant as required. It is also obvious that it satisfies $\bar{\pi}_0(\partial P) = \partial\bar{\pi}_0 P$ and, since $\hat{I}$ is generated by $\partial - X - \bar{X}$ which is mapped to 0 by $\bar{\pi}_0$, that $\hat{I} \subset \ker\bar{\pi}_0$. It remains to show that $P - \bar{\pi}_0 P \in \hat{I}$, which is equivalent to the property that $\bar{\pi}_0^2 = \bar{\pi}_0$. This however is obvious since $\bar{\pi}_0$ leaves both ∂ and $X - \bar{X}$ invariant.

The fact that $\bar{\pi}(\boxed{0\,|\,0}) = \boxed{0\,|\,0}$ is an immediate consequence of the fact that $\bar{\pi}1 = 1$. □

Proposition 6.7 *For every* $\tau \in \mathcal{S}_{\text{both}}$, *one has* $\hat{\varphi}_{\text{geo}}(\tau) \in \operatorname{range}\varphi^{\blacktriangle}_{\text{Itô}}$.

Proof. The proof of this is essentially the same as that of the converse direction for Proposition 6.4. Indeed, let $\tau \in \mathcal{S}_{\text{both}}$, for any choice of Γ, any pair $\sigma, \bar{\sigma}$ such that (3.2) holds, and any smooth family $(\psi)_{t\geq 0}$ of diffeomorphisms of $\mathbf{R}^d$ as in (6.1), one has

$$\psi_t \bullet (\Upsilon_{\Gamma,\bar{\sigma}} - \Upsilon_{\Gamma,\sigma})\tau = (\Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\bar{\sigma}} - \Upsilon_{\psi_t\bullet\Gamma,\psi_t\bullet\sigma})\tau \;.$$

Differentiating this identity at $t = 0$ and using (6.4) and (6.6), we obtain

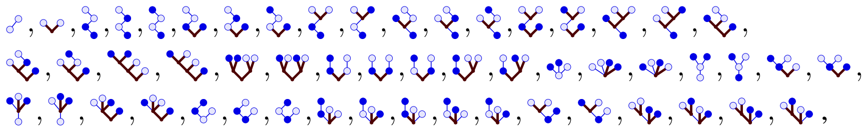
$$\Upsilon^h_{\Gamma,\sigma}\hat{\varphi}_{\text{geo}}(\tau) = \Upsilon^h_{\Gamma,\bar{\sigma}}\hat{\varphi}_{\text{geo}}(\tau).$$

Therefore, $\hat{\varphi}_{\text{geo}}(\tau) \in \mathcal{S}^{\blacktriangle}_{\text{Itô}}$ and we conclude by the fact $\mathcal{S}^{\blacktriangle}_{\text{Itô}} = \mathcal{S}_{\blacktriangle} \cap \operatorname{range}\varphi^{\blacktriangle}_{\text{Itô}}$ by Proposition 6.4. □

6.3 Dimension counting

Proposition 6.8 *The space* $\mathcal{S}$ *is of dimension* $\dim\mathcal{S} = 54$.

Proof. The proof goes by inspection. By definition, elements of $\mathcal{S}$ are given by trees $\mathfrak{S}_\circ$ (see the list of symbols on page 16), together with a pairing of their noises, modulo natural equivalence. For example, there are three inequivalent pairings for , given by , and , while all pairings of are equivalent to . The full list of canonical basis vectors obtained in this way is given by



which is indeed of cardinality 54. □

In order to obtain more information about the space $\mathcal{S}_{\mathrm{geo}}$, the following result will be useful. This is simply the algebraic counterpart of the fact that covariant differentiation transforms in the 'correct' way under the action of the diffeomorphism group.

Lemma 6.9 *Define* $\nabla\colon \mathcal{S}_0^1 \times \mathcal{S}_0^1 \to \mathcal{S}_0^1$ *by*

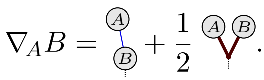
$$\nabla_A B = A \curvearrowright B + \frac{1}{2}\operatorname{tr}^2(\odot \bullet A \bullet B)\;. \tag{6.17}$$

Then, ∇ *preserves* $\ker \hat\varphi_{\mathrm{geo}}$.

Proof. Take $A, B \in \ker\hat\varphi_{\mathrm{geo}}$ so that for example $\varphi_{\mathrm{geo}}(A) = [A, \blacktriangle]$. Combining this with the defining property of an infinitesimal morphism, one has the identity

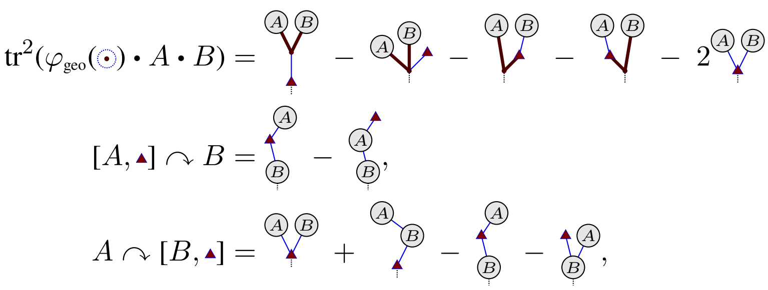
$$\begin{aligned}\varphi_{\mathrm{geo}}(\nabla_A B) &= [A,\blacktriangle] \curvearrowright B + A \curvearrowright [B,\blacktriangle] + \frac{1}{2}\operatorname{tr}^2(\varphi_{\mathrm{geo}}(\odot)\bullet A \bullet B)\\ &\quad + \frac{1}{2}\operatorname{tr}^2(\odot \bullet [A,\blacktriangle]\bullet B) + \frac{1}{2}\operatorname{tr}^2(\odot\bullet A\bullet[B,\blacktriangle])\;.\end{aligned}\tag{6.18}$$

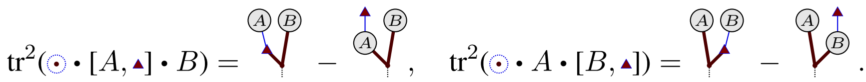
Using the diagrammatic representation (in the universal T-algebra of $\mathfrak{X}$-graphs generated by A, B and $\odot$), we can rewrite (6.17) as



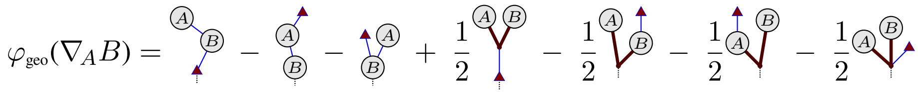
(6.19)

With this representation, (6.18) yields the identities





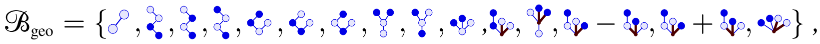
Combining these terms, we obtain



$$= [\nabla_A B, \blacktriangle]\;,$$

as claimed. □

Proposition 6.10 *The space $\mathcal{S}_{\mathrm{geo}}$ has dimension at least 15 and the set*



viewed as linear functionals on $\mathcal{S}$ is linearly independent over $\mathcal{S}_{\mathrm{geo}}$. Besides one has $\frac{1}{2}$ − − $\frac{1}{2}$ $\perp \mathcal{S}_{\mathrm{geo}}$. Furthermore, $\mathcal{S}_{\mathrm{geo}}^{\mathrm{nice}}$ is of dimension at least 13 and $\mathcal{B}_{\mathrm{geo}} \setminus \{\ ,\ \}$ is linearly independent over $\mathcal{S}_{\mathrm{geo}}^{\mathrm{nice}}$.

Proof. Write $\mathcal{S}_{\mathrm{flat}} \subset \mathcal{S}$ for the subspace generated by those trees that contain only noise nodes (so that for example $\in \mathcal{S}_{\mathrm{flat}}$ but $\notin \mathcal{S}_{\mathrm{flat}}$) and write $\pi_{\mathrm{flat}} \colon \mathcal{S} \to \mathcal{S}_{\mathrm{flat}}$ for the corresponding orthogonal projection. We similarly define $\mathcal{S}_{\mathrm{flat}}^{\circ} \subset \mathcal{S}_{\circ}$ and $\pi_{\mathrm{flat}}^{\circ} \colon \mathcal{S}_{\circ} \to \mathcal{S}_{\mathrm{flat}}^{\circ}$. It then follows immediately from (6.17) and the definition of $\curvearrowright$ that on $\mathcal{S}_{\circ}$ one has the identity

$$\pi_{\mathrm{flat}}^{\circ}(\nabla_\tau \sigma) = \pi_{\mathrm{flat}}^{\circ}\tau \curvearrowright \pi_{\mathrm{flat}}^{\circ}\sigma\;, \tag{6.20}$$

which is nothing but the algebraic counterpart of the fact that $(\nabla_X Y)^\alpha = X^\beta D_\beta Y^\alpha$ in flat space. Since $\mathcal{S}_{\mathrm{flat}}^{\circ}$ is freely generated by the elements $\circ_i$ and the operation $\curvearrowright$ (see for example [CL01]) and since ∇ preserves $\mathcal{S}_{\mathrm{geo}}^{\circ}$ by Lemma 6.9, we can define a linear map $\varphi^\circ \colon \mathcal{S}_{\mathrm{flat}}^{\circ} \to \mathcal{S}_{\mathrm{geo}}^{\circ}$ uniquely by the properties

$$\varphi^\circ(\circ_i) = \circ_i\;,\quad \varphi^\circ(\tau \curvearrowright \sigma) = \nabla_{\varphi^\circ(\tau)}\varphi^\circ(\sigma)\;.$$

By (6.20), φ° is a right inverse for $\pi_{\mathrm{flat}}^{\circ}$ and for $\tau \in \mathcal{S}_{\mathrm{flat}}^{\circ}$ one has

$$\langle \tau, \varphi^\circ(\tau)\rangle = \langle \pi_{\mathrm{flat}}^{\circ}\tau, \varphi^\circ(\tau)\rangle = \langle \tau, \pi_{\mathrm{flat}}^{\circ}\varphi^\circ(\tau)\rangle = \|\tau\|^2\;.$$

Since furthermore φ° maps $\mathcal{S}_{\mathrm{flat}}$ into $\mathcal{S}_{\mathrm{geo}}$ (when viewed as subspaces of $\mathcal{S}_{\circ}$ via ι defined in (2.9)), it follows that $\mathcal{S}_{\mathrm{flat}}$ is linearly independent over $\mathcal{S}_{\mathrm{geo}}$.

Since $\mathcal{S}_{\mathrm{flat}} = \{\ldots\}$, it only remains to find a subspace $\mathcal{S}_R \subset \mathcal{S}_{\mathrm{geo}}$ that is annihilated by $\mathcal{S}_{\mathrm{flat}}$ and such that the remaining five elements are linearly independent over $\mathcal{S}_R$. If we look at expressions involving the curvature R, then they are annihilated by $\mathcal{S}_{\mathrm{flat}}$ (since $R = 0$ whenever $\Gamma = 0$, but this is also seen easily from the algebraic representation of R since it only involves trees with at least one Γ-node). Then, we first notice that and are free over $\nabla_{\circ}(R(\circ,\bullet)\bullet)$ and $\nabla_{R(\circ,\bullet)\bullet}\circ$. Indeed, since one has

$$R(\circ,\bullet)\bullet = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} ,$$

the expression (6.19) of the covariant derivative shows that annihilates $\nabla_{\circ}(R(\circ,\bullet)\bullet)$ but not $\nabla_{R(\circ,\bullet)\bullet}\circ$, while the opposite is true for .

Then the remaining terms , $-$, $+$ are free over $R(\circ,\nabla_{\bullet}\bullet)\circ$, $R(\circ,\nabla_{\bullet}\circ)\bullet$ and $R(\circ,\nabla_{\bullet}\circ - 2\nabla_{\circ}\bullet)\bullet$, given by

$$R(\circ,\nabla_{\bullet}\circ)\bullet = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} + \frac{1}{8} - \frac{1}{4} - \frac{1}{8}$$
$$R(\circ,\nabla_{\circ}\bullet)\bullet = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} + \frac{1}{8} - \frac{1}{4} - \frac{1}{8}$$
$$R(\circ,\nabla_{\bullet}\bullet)\circ = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} + \frac{1}{8} - \frac{1}{4} - \frac{1}{8}$$

which are all three annihilated by and . The last statement follows from Corollary 3.19.

It remains to prove that $\frac{1}{2} - - \frac{1}{2} \perp \mathcal{S}_{\text{geo}}$. For any $\tau \in \mathcal{S}_{\text{geo}}$, we have $\langle \hat\varphi_{\text{geo}}(\tau), \rangle = 0$, so that $\langle \tau, \hat\varphi^*_{\text{geo}}()\rangle = 0$. We conclude from the fact that $\hat\varphi^*_{\text{geo}}() = \frac{1}{2} - - \frac{1}{2}$ as a consequence of a simple calculation, using the fact that these three basis vectors are the only ones that can possibly generate a copy of , as well as the fact that $||^2 = ||^2 = 2$ while $||^2 = 4$. $\square$

Proposition 6.11 *One has* $\mathcal{B}_{\text{Itô}} \subset \mathcal{S}_{\text{Itô}}$ *with*

$$\mathcal{B}_{\text{Itô}} = \left\{ \begin{array}{l} , , + , , , , , , + \\ , , , , , , , , + , \end{array} \right\} .$$

In particular, $\mathcal{S}_{\text{Itô}}$ *is of dimension at least* 19.

Proof. Each of these elements belongs to the image of $\varphi_{\text{Itô}}$ by inspection. For example, $\varphi_{\text{Itô}}(\frac{1}{2}) = +$ and $\varphi_{\text{Itô}}() = $. $\square$

Corollary 6.12 *One has* $\dim(\mathcal{S}_{\text{geo}} + \mathcal{S}_{\text{Itô}}) \geq |\mathcal{B}_{\text{geo}}| + |\mathcal{B}_{\text{Itô}}| - 2 = 32$.

Proof. It suffices to note that all distinct elements of $\mathcal{B}_{\text{geo}} \cup \mathcal{B}_{\text{Itô}}$ are mutually orthogonal and that $|\mathcal{B}_{\text{geo}} \cap \mathcal{B}_{\text{Itô}}| = 2$. $\square$

Let us define the free T-algebra $\hat{\mathcal{S}}_{\bullet,\blacktriangle}$ in the same way as $\hat{\mathcal{S}}_{\bullet}$, except that we add an element $\blacktriangle$ of degree $(1,0)$ the set of generators. We define the morphism $\psi_{\blacktriangle}\colon \hat{\mathcal{S}}_{\bullet,\blacktriangle} \to \bar{\mathcal{S}}_{\circ,\blacktriangle}$ to be the unique extension of the morphism ψ defined in (6.8) that sends $\blacktriangle$ to $\blacktriangle$ and we set $\bar{\mathcal{S}}_{\bullet,\blacktriangle} = \hat{\mathcal{S}}_{\bullet,\blacktriangle}/\ker\psi_{\blacktriangle}$. Note that similarly to before, both $\mathcal{S}$ and $\mathcal{S}_{\blacktriangle}$ can (and will from now on) be seen as subspaces of $\bar{\mathcal{S}}_{\bullet,\blacktriangle}$ and $\varphi^{\blacktriangle}_{\text{Itô}}$ can be viewed as the map $\varphi^{\blacktriangle}_{\text{Itô}}\colon \bar{\mathcal{S}}_{\square,\blacktriangle} \to \bar{\mathcal{S}}_{\bullet,\blacktriangle}$ sending $\odot$ to $\odot$, $\square$ to $\boxed{0|0}$ and $\blacktriangle$ to $\blacktriangle$.

Let $\hat m_{\text{Itô}} : \hat{\mathcal S}_{\bullet,\blacktriangle} \to \bar{\mathcal S}_{\square,\blacktriangle}$ be the unique morphism of T-algebras such that, for $(k,\ell) \neq (0,0)$ we have

$$\hat m_{\text{Itô}}(\blacktriangle) = \blacktriangle\;, \quad \hat m_{\text{Itô}}(\odot) = \odot\;, \quad \hat m_{\text{Itô}}(\boxed{0|0}) = \square\;, \quad \hat m_{\text{Itô}}(\boxed{k|\ell}) = 0\;.$$

By Lemma 6.6, there exists a right inverse $\iota^\blacktriangle : \bar{\mathcal S}_{\bullet,\blacktriangle} \to \hat{\mathcal S}_{\bullet,\blacktriangle}$ for the canonical projection $\pi^\blacktriangle : \hat{\mathcal S}_{\bullet,\blacktriangle} \to \bar{\mathcal S}_{\bullet,\blacktriangle}$. Using this, we then define $m_{\text{Itô}} : \bar{\mathcal S}_{\bullet,\blacktriangle} \to \bar{\mathcal S}_{\square,\blacktriangle}$ by

$$m_{\text{Itô}} = \hat m_{\text{Itô}} \circ \iota^\blacktriangle\;.$$

Note that since $\iota^\blacktriangle \pi^\blacktriangle \boxed{0|0} = \boxed{0|0}$ by Lemma 6.6, it follows immediately that

$$m_{\text{Itô}} \circ \varphi^\blacktriangle_{\text{Itô}} = \mathrm{id} \colon \bar{\mathcal S}_{\square,\blacktriangle} \to \bar{\mathcal S}_{\square,\blacktriangle}\;. \tag{6.21}$$

In order to see how $m_{\text{Itô}}$ (and therefore also $P_{\text{Itô}}$ defined in (6.22) below) acts in a more concrete way, note that by (6.10), an element in $\bar{\mathcal S}_{\bullet,\blacktriangle}$ can always be written in terms of the generators $\boxed{k|\ell}$ without having their derivatives appear. Furthermore, by the correspondence (6.15) and the explicit formula (6.16), $\iota^\blacktriangle$ maps $\boxed{k|\ell}$ to $2^{-k-\ell}\partial^{k+\ell}\boxed{0|0}$, which is then mapped to $2^{-k-\ell}\partial^{k+\ell}\square$ by $\hat m_{\text{Itô}}$. For example, it follows that one has

$$m_{\text{Itô}}(\;) = \frac{1}{4}\;, \qquad m_{\text{Itô}}(\;) = \frac{1}{4}\;.$$

Note here that there is a major difference between these two examples. In the first case, the $\mathfrak X$-graph , viewed as a directed graph between its vertices, is acyclic, while in the second case it contains a cycle. As a consequence, the morphism $\varphi_{\text{Itô}}$ maps the first graph back into $\mathcal S$, while the second graph is mapped to an element of $\bar{\mathcal S}$ that does not belong to to $\mathcal S$.

Remark 6.13 Elements in $\bar{\mathcal S}_{\bullet,\blacktriangle}$ can naturally be represented by objects of the type $(V_g, \mathfrak t, \varphi, \mathcal P_g)^u_\ell$, where $(V_g, \mathfrak t, \varphi)^u_\ell$ is an $\mathfrak X$-graph for $\mathfrak X = \{\odot, \blacktriangle, \circ\}$ such that there are an even number of vertices in V_g of type $\circ$ and $\mathcal P_g$ is a pairing of these vertices. The notion of isomorphism is also the same as for $\mathfrak X$-graphs except that of course pairings need to be preserved (but the two elements within a pair can be exchanged since these pairs are unordered). This is consistent with our existing graphical notation for elements of $\mathcal S$ which are represented as graphs together with a pairing of the vertices of type $\circ$. The existing scalar product on $\mathcal S$ then coincides with the natural scalar product on $\bar{\mathcal S}_{\bullet,\blacktriangle}$ for which $\langle g, g\rangle$ equals the number of its automorphisms.

One also has representations of the adjoints of multiplication, trace and derivation that are essentially the same as in Section 5.4 with the obvious modifications. (In particular the operator Δ is not allowed to split pairs.)

This motivates the introduction of the linear map $P^{\text{acyc}} : \bar{\mathcal S}_{\bullet,\blacktriangle} \to \bar{\mathcal S}_{\bullet,\blacktriangle}$ which maps every acyclic $\mathfrak X$-graph in $\bar{\mathcal S}_{\bullet,\blacktriangle}$ to itself, while it maps those $\mathfrak X$-graphs containing a cycle to 0. A very important remark here is that *in the notion of 'acyclic graph', paired nodes should be identified* so that is considered acyclic while is not. Note also that P^{acyc} is of course not a morphism of T-algebras and that it is well-defined since the quotienting procedure for $\odot$ does not affect the directed graph structure.

With these definitions in place, we set

$$P_{\text{Itô}} = P^{\text{acyc}} \circ M_{\text{Itô}} \colon \bar{\mathcal{S}}_{\bullet,\blacktriangle} \to \bar{\mathcal{S}}_{\bullet,\blacktriangle}\;, \tag{6.22}$$

where $M_{\text{Itô}} = \varphi^{\blacktriangle}_{\text{Itô}} \circ m_{\text{Itô}}$, so that one has for example $P_{\text{Itô}}(\;) = \frac{1}{2}\; + \frac{1}{2}\;$, while $P_{\text{Itô}}(\;) = 0$. Let us remark that, since in the notion of 'acyclic graph', paired nodes should be identified, one can define a similar projection on $\bar{\mathcal{S}}_{\square,\blacktriangle}$, also denoted by P^{acyc}, such that $P_{\text{Itô}} = \varphi^{\blacktriangle}_{\text{Itô}} \circ P^{\text{acyc}} \circ m_{\text{Itô}}$: one can disregard non-acyclic graphs either before or after applying $\varphi^{\blacktriangle}_{\text{Itô}}$.

Let $\mathrm{Vec}(\mathcal{S}, \mathcal{S}_{\blacktriangle})$ be the vector space generated by $\mathcal{S}$ and $\mathcal{S}_{\blacktriangle}$. As a consequence of the previous properties, one can deduce the following lemma:

Lemma 6.14 *The linear map $P_{\text{Itô}}$ is a self-adjoint projection on* $\mathrm{Vec}(\mathcal{S}, \mathcal{S}_{\blacktriangle})$ *and* $\mathrm{range}(P_{\text{Itô}}) = \mathrm{Vec}(\mathcal{S}_{\text{Itô}}, \mathcal{S}^{\blacktriangle}_{\text{Itô}})$.

Proof. For $m \geq 0$, write $E_m \subset \bar{\mathcal{S}}_{\bullet,\blacktriangle}$ for the set of elements of the form $\alpha\boxed{k}\boxed{\ell}$ for $k + \ell = m$ and $\alpha \in \mathrm{Sym}(m, 2)$. This set is of cardinality 2^m since by the correspondence indicated in Remark 6.13 we can interpret its elements as precisely those graphs with m inputs, two outputs, and two nodes of type $\circ$, so that that there are 2^m ways of connecting the inputs to the two nodes. (There are of course also two ways of connecting these nodes to the two outputs, but these are isomorphic to the graphs obtained by a suitable permutation of the inputs.) Furthermore, since all of their edges are anchored,

The map $M_{\text{Itô}}$ is then given by

$$M_{\text{Itô}} h = 2^{-m} \sum_{g \in E_m} g\;, \qquad \forall h \in E_m\;.$$

Since all elements of E_m have the same directed graph structure (since for the purpose of that structure we identify the two vertices of type $\circ$), it follows that $M_{\text{Itô}}$ does not change that structure. This immediately shows that $M_{\text{Itô}}$ commutes with P^{acyc}, so that $P^2_{\text{Itô}} = P_{\text{Itô}}$, since both P^{acyc} and $M_{\text{Itô}}$ are idempotent. To show that $P_{\text{Itô}}$ is self-adjoint, it therefore suffices to show that both P^{acyc} and $M_{\text{Itô}}$ are self-adjoint. The fact that this is the case for P^{acyc} is obvious since it is diagonal in the basis given by $\mathcal{X}$-graphs with pairings, which is also orthogonal for our scalar product.

Since elements of E_m have no internal symmetries, they are orthonormal, which immediately implies that $M_{\text{Itô}} = M^*_{\text{Itô}}$ on $\bigoplus_{m\geq 0} \langle E_m \rangle$. (As a matrix, it is given on each $\langle E_m \rangle$ by the matrix with all entries identical and equal to 2^{-m}.) Since one also has $M^*\odot = M\odot$ and $M^*\blacktriangle = M\blacktriangle$, it remains to show that $M^*_{\text{Itô}}$ is a morphism of T-algebras to then conclude that $M^* = M$. For this, we need to show that $M^*_{\text{Itô}}$ commutes with the four defining operations of a T-algebra, namely the action of the symmetric group, multiplication, trace and derivation. Equivalently, we need to show that $M_{\text{Itô}}$ commutes with the adjoints of these three operations, as described in Section 5.4 and Remark 6.13.

The fact that $(M_{\text{Itô}} \otimes M_{\text{Itô}})\Delta\tau = \Delta M_{\text{Itô}}\tau$ follows immediately from the fact that, since it does not change the directed graph structure, $M_{\text{Itô}}$ maps irreducible elements to irreducible elements. Similarly, the fact that $M_{\text{Itô}}\alpha^*\tau = \alpha^* M_{\text{Itô}}\tau$ is immediate from the fact that $\alpha^* = \alpha^{-1}$ and $M_{\text{Itô}}$ itself is a morphism of T-algebras.

We now show that $M_{\text{Itô}}\,\mathrm{tr}^*\, g = \mathrm{tr}^*\, M_{\text{Itô}} g$ for any $\mathfrak{X}$-graph $g = (V_g, \mathfrak{t}, \varphi, \mathscr{P}_g)^u_\ell \in \bar{\mathcal{S}}_{\bullet,\blacktriangle}$. Note that $M_{\text{Itô}}g$ is the average over all graphs of the type $\hat{g} = (V_g, \mathfrak{t}, \hat{\varphi}, \mathscr{P}_g)^u_\ell$ where, whenever $\varphi(e) = (v,\star)$ for some v of type $\circ$, one has $\hat{\varphi}(e) \in \{(v,\star),(\bar{v},\star)\}$, where $\bar{v}$ is the unique vertex such that $\{v,\bar{v}\} \in \mathscr{P}_g$. If $\varphi(e)$ is not of this type, then one has $\hat{\varphi}(e) = \varphi(e)$. Note also that for any of the graphs $\hat{g}$ appearing in the description of $M_{\text{Itô}}g$, one has a natural identification of $\mathrm{Out}(\hat{g})$ with $\mathrm{Out}(g)$ and, for any $\{v,\bar{v}\} \in \mathscr{P}_g$. it is still the case that if $\mathrm{cod}(e) \in \{v,\bar{v}\}$ in g, then the same is true in $\hat{g}$. It is then straightforward to see that

$$\mathrm{Cut}_e\, M_{\text{Itô}} g = M_{\text{Itô}}\,\mathrm{Cut}_e\, g\;,$$

which immediately implies the claimed commutation.

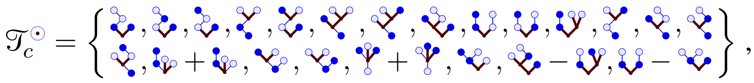
The fact that $M_{\text{Itô}}\partial^* g = \partial^*(M_{\text{Itô}}g)$ follows in a similar way, thus completing the proof that $M_{\text{Itô}}$ is self-adjoint.

It remains to show that, when restricted to $\mathrm{Vec}(\mathcal{S},\mathcal{S}_\blacktriangle)$, $P_{\text{Itô}}$ is indeed the projection onto $\mathrm{Vec}(\mathcal{S}_{\text{Itô}},\mathcal{S}^\blacktriangle_{\text{Itô}})$. Recall that $\mathrm{Vec}(\mathcal{S},\mathcal{S}_\blacktriangle)$ is spanned by the irreducible graphs of degree $(1,0)$ with at most two pairs of vertices of type $\circ$, which are furthermore connected when viewed as $\mathfrak{X}$-graphs *without* identifying the paired vertices. The image of the map $M_{\text{Itô}}$ is *not* in general contained in this space since, although it preserves the directed graph structure with pairs of vertices identified, it does not preserve the structure without this identification. We claim however that $M_{\text{Itô}}$ maps the image of P^{acyc} into $\mathrm{Vec}(\mathcal{S},\mathcal{S}_\blacktriangle)$. This is because, since we consider only objects of type $(1,0)$, the only way the graph obtained without identifying paired vertices can be disconnected is if it has a connected component of type $(0,0)$. Since every generator has exactly one output, such a component has necessarily a cycle and therefore cannot appear in the image of P^{acyc}.

The fact that $\mathrm{range}(P_{\text{Itô}}) = \mathrm{Vec}(\mathcal{S}_{\text{Itô}},\mathcal{S}^\blacktriangle_{\text{Itô}})$ then follows from Proposition 6.4, noting that since $\varphi^\blacktriangle_{\text{Itô}}$ is injective and it maps cyclic graphs out of $\mathrm{Vec}(\mathcal{S},\mathcal{S}_\blacktriangle)$, restricting its domain to acyclic graphs does not affect the intersection of its image with $\mathrm{Vec}(\mathcal{S}_{\text{Itô}},\mathcal{S}^\blacktriangle_{\text{Itô}})$. □

Theorem 6.15 *One has* $\mathcal{S}_{\text{Itô}} + \mathcal{S}_{\text{geo}} = \mathcal{S}_{\text{both}}$ *and* $\mathcal{S}_{\text{Itô}} \cap \mathcal{S}_{\text{geo}} = \langle\{\tau_\star,\tau_c\}\rangle$. *Furthermore,* $\mathcal{S}_{\text{Itô}} = \langle\mathscr{B}_{\text{Itô}}\rangle$.

Proof. Since $\mathcal{S}_{\text{Itô}} + \mathcal{S}_{\text{geo}} \subset \mathcal{S}_{\text{both}}$ and using Corollary 6.12, it suffices to show that $\dim\mathcal{S}_{\text{both}} \leq 32$ in order to prove that $\mathcal{S}_{\text{Itô}} + \mathcal{S}_{\text{geo}} = \mathcal{S}_{\text{both}}$ and $\dim(\mathcal{S}_{\text{Itô}} \cap \mathcal{S}_{\text{geo}}) = 2$. Since $\dim\mathcal{S} = 54$, this means that it suffices to find a collection $\mathscr{E}^*$ of 22 linearly independent elements of $\mathcal{S}^*$ such that $\mathcal{S}_{\text{both}} \subset \bigcap_{e\in\mathscr{E}^*}\ker(e)$. We claim that a possible choice for $\mathscr{E}^*$ is given by the linear functionals determined by $\mathscr{T}_c^{\odot} \subset \mathcal{S}$ (using the correspondence $\mathcal{S} \approx \mathcal{S}^*$ given by the scalar product) given by



which has cardinality 22.

For all $\tau \in \mathcal{T}_c^{\odot}$, except the two last elements, $m_{\text{Itô}}(\tau)$ is a linear combination of cyclic graphs, while for the two last elements, $m_{\text{Itô}}(\tau) = 0$, so that $P_{\text{Itô}}(\tau) = 0$ for all $\tau \in \mathcal{T}_c^{\odot}$.

Let $\varphi_{\blacktriangle} : \bar{\mathcal{S}}_{\bullet} \to \bar{\mathcal{S}}_{\bullet,\blacktriangle}$ be the *infinitesimal* morphism of T-algebras with respect to the canonical injection, mapping $\odot$ to $\partial^2\blacktriangle$ and $\boxed{k|l}$ to 0 (this is well-defined since $S^1_{1,1}\partial^2\blacktriangle = \partial^2\blacktriangle$ by (5.6) and since all other identifications in (6.9) and (6.10) involve linear combinations of the generators $\boxed{k|\ell}$). Since $P_{\text{Itô}} \circ \varphi_{\blacktriangle} = \varphi_{\blacktriangle} \circ P_{\text{Itô}}$ as a consequence of the fact that $\varphi_{\blacktriangle}$ maps any $\mathcal{X}$-graph to a linear combination of $\mathcal{X}$-graphs having the same directed graph structure, we observe that for any $\tau \in \mathcal{T}_c^{\odot}$, $\varphi_{\blacktriangle}(\tau) \in \ker(P_{\text{Itô}})$.

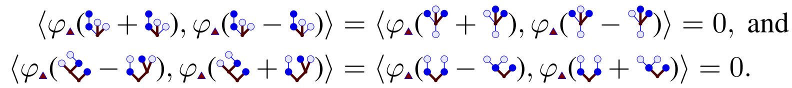
Using Proposition 6.7 and Lemma 6.14, for any $v \in \mathcal{S}_{\text{both}}$, $\hat{\varphi}_{\text{geo}}(v) \in \operatorname{range} P_{\text{Itô}}$, and, since $P_{\text{Itô}}$ is self-adjoint, $\hat{\varphi}_{\text{geo}}(v)$ is orthogonal to $\ker P_{\text{Itô}}$. Thus for $\tau \in \mathcal{T}_c^{\odot}$, $\langle \varphi_{\blacktriangle}(\tau), \hat{\varphi}_{\text{geo}}(v)\rangle = 0$, hence $\langle \hat{\varphi}^*_{\text{geo}}\varphi_{\blacktriangle}(\tau), v\rangle = 0$. From now on, let us denote $\ell_\tau = \hat{\varphi}^*_{\text{geo}}\varphi_{\blacktriangle}(\tau)$; it remains to prove that $\{\ell_\tau : \tau \in \mathcal{T}_c^{\odot}\} \subset \mathcal{S}^*$ is a linearly independent family. Let us remark that we omitted $-$ in the definition of $\mathcal{T}_c^{\odot}$ since $\varphi_{\blacktriangle}(-) = 0$.

For any tree τ in $\mathcal{S}_{\blacktriangle}$, we denote by $\mathsf{dg}(\tau)$ the number of $\odot$ and $\blacktriangle$ vertices in τ. For any non-zero element $\eta \in \mathcal{S}_{\blacktriangle}$, if d is an integer such that $\langle \eta, \tau\rangle = 0$ as soon as $\mathsf{dg}(\tau) \neq d$, then we define $\mathsf{dg}(\eta) = d$. Using (6.2), one gets that for any $\tau \in \mathcal{S}$ such that $\hat{\varphi}_{\text{geo}}(\tau) + 2\varphi_{\blacktriangle}(\tau) \neq 0$, $\mathsf{dg}(\hat{\varphi}_{\text{geo}}(\tau) + 2\varphi_{\blacktriangle}(\tau)) = \mathsf{dg}(\tau) + 1$. In particular, since trees of different degrees are orthogonal, for any τ and $\tilde{\tau}$ such that $\mathsf{dg}(\tilde{\tau}) \geq \mathsf{dg}(\tau)$,

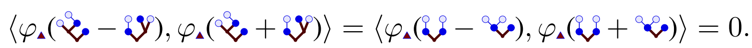
$$\langle \ell_\tau, \tilde{\tau}\rangle = \langle \varphi_{\blacktriangle}(\tau), \hat{\varphi}_{\text{geo}}(\tilde{\tau})\rangle = -2\langle \varphi_{\blacktriangle}(\tau), \varphi_{\blacktriangle}(\tilde{\tau})\rangle. \tag{6.23}$$

In particular, this proves that for any tree τ in $\mathcal{T}_c^{\odot}$, any $\tilde{\tau}$ in $\mathcal{S}$ such that $\mathsf{dg}(\tilde{\tau}) \geq \mathsf{dg}(\tau)$, $\langle \ell_\tau, \tilde{\tau}\rangle = 0$ and $\langle \ell_\tau, \tau\rangle \neq 0$. We will show that this holds also for the remaining elements τ in $\{ + , + , - , - \} \subset \mathcal{T}_c^{\odot}$.

Instead of working with the usual basis of $\mathcal{S}$ which consists of the usual trees, let us change the family $\{ , , , , , , , \}$ into the family $\{ + , - , + , - , - , + , - , + \}$. Using this basis, it remains to prove that:



The first assertion is a consequence of the fact that $\varphi_{\blacktriangle}(-) = 0 = \varphi_{\blacktriangle}(-)$. As for the second, it can be deduced from the fact that for any τ appearing in $\varphi_{\blacktriangle}(+)$, $\|\tau\| = 1$ and for any τ appearing in $\varphi_{\blacktriangle}(+)$, $\|\tau\|^2 = 2$, one has



Using (6.23), we have proven that for any $\tau \in \mathcal{T}_c^{\odot}$, any element $\tilde{\tau}$ of this new basis such that $\mathrm{dg}(\tilde{\tau}) \geq \mathrm{dg}(\tau)$, $\langle \ell_\tau, \tilde{\tau} \rangle \neq 0$ if and only if $\tau = \tilde{\tau}$. In particular, this proves that $\ell_\tau = \alpha_\tau \tau + \sigma$ with $\mathrm{dg}(\sigma) < \mathrm{dg}(\tau)$ and α_τ a non-zero real: the elements $\{\ell_\tau : \tau \in \mathcal{T}_c^{\odot}\} \subset \mathcal{S}^*$ are linearly independent thanks to the triangular structure given by counting the number of $\odot$.

Let us study the intersection of $\mathcal{S}_{\text{Itô}}$ and $\mathcal{S}_{\text{geo}}$. Using the previous results, we get that $\dim(\mathcal{S}) = 54$ and also $\dim(\mathcal{S}_{\text{Itô}} \cap \mathcal{S}_{\text{geo}}) = 2$. In order to conclude, it remains to prove that $\tau_\star, \tau_c \subset \mathcal{S}_{\text{Itô}} \cap \mathcal{S}_{\text{geo}}$. Using Lemma 6.9 and (3.20), (3.21), we get that $\tau_\star, \tau_c \in \mathcal{S}_{\text{geo}}$. Besides, one can check that $\tau_\star, \tau_c \in \mathcal{S}_{\text{Itô}}$ using the explicit formulas (3.22) and (3.23). $\square$

Remark 6.16 This theorem implies that $\dim \mathcal{S}_{\text{geo}} = 15$ and that the family

$$(\varphi^{\circ}(\tau))_{\tau \in \mathcal{B}_{\text{geo}} \cap \mathcal{S}_{\text{flat}}} \cup \{\nabla_\circ(R(\circ,\bullet)\bullet),\ \nabla_{R(\circ,\bullet)\bullet}\circ,\ R(\circ,\nabla_\bullet\bullet)\circ,\ R(\circ,\nabla_\bullet\circ)\bullet,\ R(\circ,\nabla_\bullet\circ - 2\nabla_\circ\bullet)\bullet\}$$

provided by the proof of Proposition 6.11 is a basis of $\mathcal{S}_{\text{geo}}$. After a change of basis, one can verify that a simpler one is given by $\mathfrak{V} \cup \{\nabla_\circ\circ\}$, where $\mathfrak{V}$ was defined in (1.8). In particular, since $\Upsilon_{\Gamma,\sigma}(\nabla_\tau\bar{\tau}) = \nabla_{\Upsilon_{\Gamma,\sigma}\tau}\Upsilon_{\Gamma,\sigma}\bar{\tau}$, this proves that for any $\tau \in \mathcal{S}_{\text{geo}}$, the identity (3.3) automatically holds for all diffeomorphisms. Let us remark that the only element missing from the list of combinatorial covariant derivatives is $\nabla_\bullet\nabla_\circ\nabla_\bullet\circ$, the reason being that it is equal to

$$\nabla_\circ\nabla_\bullet\nabla_\bullet\circ + \nabla_\bullet\nabla_{\nabla_\bullet\circ}\circ - \nabla_{\nabla_\bullet\nabla_\bullet\circ}\circ - \nabla_{\nabla_{\nabla_\bullet\circ}\circ}\bullet + \nabla_{\nabla_{\nabla_\bullet\circ}\bullet}\circ - \nabla_\circ\nabla_{\nabla_\bullet\circ}\bullet + \nabla_{\nabla_\circ\nabla_\bullet\circ}\bullet. \tag{6.24}$$

Finally, we conclude from the previous theorem that $\dim(\mathcal{S}_{\text{Itô}}) = 19$ and that $\mathcal{B}_{\text{Itô}}$ is a basis of $\mathcal{S}_{\text{Itô}}$.

Appendix A Symbolic index

In this appendix, we collect the most used symbols of the article, together with their meaning and the page where they were first introduced.

Symbol	Meaning	Page
$[u]$	Set $\{i \in \mathbf{N} : 1 \leq i \leq u\}$	47
$\mathcal{B}_{\text{geo}}$	Linear independent set over $\mathcal{S}_{\text{geo}}$	72
$\mathcal{B}_{\text{Itô}}$	Collection of trees generating $\mathcal{S}_{\text{Itô}}$	73
$\mathcal{B}_\star^a$	Space in which the laws of the solutions take values	7
$C_{\varepsilon,\text{geo}}^{\text{BPHZ}}$	Element of $\mathcal{S}$ renormalising $U_\varepsilon^{\text{geo}}$	13
$C_{\varepsilon,\text{Itô}}^{\text{BPHZ}}$	Element of $\mathcal{S}$ renormalising $U_\varepsilon^{\text{Itô}}$	13
$\hat{C}_{\varepsilon,\text{geo}}^{\text{BPHZ}}$	Element of $\mathcal{S}_{\text{geo}}$ such that $\hat{C}_{\varepsilon,\text{geo}}^{\text{BPHZ}} - C_{\varepsilon,\text{geo}}^{\text{BPHZ}}$ converges	13
$\hat{C}_{\varepsilon,\text{Itô}}^{\text{BPHZ}}$	Element of $\mathcal{S}_{\text{Itô}}$ such that $\hat{C}_{\varepsilon,\text{Itô}}^{\text{BPHZ}} - C_{\varepsilon,\text{Itô}}^{\text{BPHZ}}$ converges	13
$\mathcal{C}_\star^a$	Space in which the solutions take values	7
∂	Abstract derivation from $\mathcal{V}_\ell^u$ to $\mathcal{V}_{\ell+1}^u$	49

Symbol	Meaning	Page
Γ	Christoffel symbols for the Levi-Civita connection	3
g	Inverse metric tensor of $\mathscr{M}$ given by $\sigma\sigma^\top$	4
$H_{\Gamma,\sigma}$	Vector field statisfying $H_{\Gamma,\sigma} = \Upsilon_{\Gamma,\sigma}\tau_\star$	9
$m_{\text{Itô}}$	Linear map from $\bar{\mathcal{S}}_{\bullet,\blacktriangle}$ to $\bar{\mathcal{S}}_{\square,\blacktriangle}$ replacing a pairing by $\square$	74
$\mathscr{M}$	Compact Riemannian manifold	2
$P_{\text{Itô}}$	Self-adjoint projection on $\mathcal{S}_\blacktriangle$, range $P_{\text{Itô}} = \mathcal{S}^\blacktriangle_{\text{Itô}}$	75
$\mathbf{\Pi}^{\text{BPHZ}}$	BPHZ model	18
$\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}}$	Model of the geometric regularisation	17
$\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}$	Model of the Itô regularisation	17
$\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{geo}}$	BPHZ renormalisation of $\mathbf{\Pi}^{(\varepsilon)}_{\text{geo}}$	18
$\hat{\mathbf{\Pi}}^{(\varepsilon)}_{\text{Itô}}$	BPHZ renormalisation of $\mathbf{\Pi}^{(\varepsilon)}_{\text{Itô}}$	18
$\mathbb{Q}$	Set of pairs $(\sigma,\bar\sigma)$ such that $\sigma\sigma^\top = \bar\sigma\bar\sigma^\top$	25
$\nabla_X Y$	Covariant derivative of Y in the direction of X	8
σ_i	Collection of vector fields on $\mathscr{M}$	4
R	Riemannian curvature tensor	9
$\mathfrak{S}_0$	Symbols with one noise type	22
$\mathfrak{S}^{(k)}_\circ$	Collection of trees with k noises	17
$\mathfrak{S}_\circ$	Collection of trees $\mathfrak{S}^{(2)}_\circ \cup \mathfrak{S}^{(4)}_\circ$	17
$\mathcal{S}_\circ$	Linear span of $\mathfrak{S}_\circ$	17
$\bar{\mathcal{S}}_\circ$	Extension of $\mathcal{S}_\circ$ with disconnected graphs and loops	63
$\hat{\mathcal{S}}_\bullet$	Free T-algebra with generators $\odot$ and $\boxed{k\mid\ell}$	65
$\hat{\mathcal{S}}_{\bullet,\blacktriangle}$	Extension of $\hat{\mathcal{S}}_\bullet$ with the generator $\blacktriangle$	73
$\bar{\mathcal{S}}_\bullet$	Quotiented space given by $\bar{\mathcal{S}}_\bullet = \hat{\mathcal{S}}_\bullet / \ker\psi$	65
$\bar{\mathcal{S}}_{\bullet,\blacktriangle}$	Quotiented space given by $\bar{\mathcal{S}}_{\bullet,\blacktriangle} = \hat{\mathcal{S}}_{\bullet,\blacktriangle} / \ker\psi_\blacktriangle$	73
$\mathcal{S}_\blacktriangle$	Subspace of $\bar{\mathcal{S}}_{\circ,\blacktriangle}$ defined by $\mathcal{S}_\blacktriangle = \varphi_{\text{geo}}(\mathcal{S})$	65
$\bar{\mathcal{S}}_{\circ,\blacktriangle}$	Extension of $\bar{\mathcal{S}}_\circ$ that includes the generator $\blacktriangle$	63
$\bar{\mathcal{S}}_\square$	Same definition as for $\bar{\mathcal{S}}_\circ$ but with $\square$ instead of the $\circ_i$	65
$\bar{\mathcal{S}}_{\square,\blacktriangle}$	Extension of $\bar{\mathcal{S}}_\square$ that includes the generator $\blacktriangle$	65
$\mathcal{S}^{\text{nice}}$	Subspace of $\mathcal{S}$ such that $\mathcal{S}^{\text{nice}} = \mathscr{V}^{\text{nice}} \cap \mathcal{S}$	13
$\mathcal{S}^{\text{nice}}_{\text{geo}}$	Subspace of $\mathcal{S}^{\text{nice}}$ such that $\mathcal{S}^{\text{nice}}_{\text{geo}} = \mathcal{S}^{\text{nice}} \cap \mathcal{S}_{\text{geo}}$	35
$\mathcal{S}^{\text{nice}}_{\text{Itô}}$	Subspace of $\mathcal{S}^{\text{nice}}$ such that $\mathcal{S}^{\text{nice}}_{\text{Itô}} = \mathcal{S}^{\text{nice}} \cap \mathcal{S}_{\text{Itô}}$	35
$\hat{\mathfrak{S}}$	Trees endowed with partition of noises into pairs	22
$\mathcal{S}$	Linear span of elements of $\hat{\mathfrak{S}}$ with at most 4 noises	22
$\mathcal{S}_{\text{both}}$	Subspace of $\mathcal{S}$ such that $\mathcal{S}_{\text{both}} = \mathcal{S}_{\text{Itô}} + \mathcal{S}_{\text{geo}}$	13
$\mathcal{S}_{\text{geo}}$	Set of geometric elements in $\mathcal{S}$	26
$\mathcal{S}_{\text{Itô}}$	Set of Itô elements in $\mathcal{S}$	26
$\mathcal{S}^\blacktriangle_{\text{Itô}}$	Subspace of Itô elements in $\mathcal{S}_\blacktriangle$	65
$S^{u_1,u_2}_{\ell_1,\ell_2}$	Element of $\mathrm{Sym}(u_1+u_2,\ell_1+\ell_2)$	48

Symbol	Meaning	Page
$\mathrm{Sym}(u)$	Symmetric group on $[u]$	47
$\mathrm{Sym}(u,\ell)$	Direct product of groups $\mathrm{Sym}(u) \times \mathrm{Sym}(\ell)$	47
$\tau_\star$	Degree of freedom of canonical family of solutions	9, 36
τ_c	Additional degree of freedom for non-canonical family	36
tr	Abstract partial trace from $\mathscr{V}_{\ell+1}^{u+1}$ to $\mathscr{V}_\ell^u$	48
$U^\flat$	Law of the canonical family of solutions	9
U^{geo}	Law of the geometric solution	6
$U^{\mathrm{Itô}}$	Law of the solution satisfying the Itô's isometry	6
U^{BPHZ}	Law of the BPHZ solution	25
$\Upsilon_{\Gamma,\sigma}$	Valuation map	20
φ_{geo}	Geometric infinitesimal morphism from $\bar{\mathcal{S}}_\circ$ to $\bar{\mathcal{S}}_{\circ,\blacktriangle}$	63
$\hat\varphi_{\mathrm{geo}}$	Linear map defined by $\varphi_{\mathrm{geo}} - [\cdot, \blacktriangle]$	64
$\varphi_{\mathrm{Itô}}$	Itô infinitesimal morphism from $\bar{\mathcal{S}}_\square$ to $\bar{\mathcal{S}}_\circ$	65
$\varphi_{\mathrm{Itô}}^{\blacktriangle}$	Itô infinitesimal morphism from $\bar{\mathcal{S}}_{\square,\blacktriangle}$ to $\bar{\mathcal{S}}_{\circ,\blacktriangle}$	65
$\mathfrak{X}$	A finite number of types	51
ψ	Morphism from $\hat{\mathcal{S}}_\bullet$ to $\bar{\mathcal{S}}_\circ$	66
$\psi_\blacktriangle$	Morphism from $\hat{\mathcal{S}}_{\bullet,\blacktriangle}$ to $\bar{\mathcal{S}}_{\circ,\blacktriangle}$	73
$\mathfrak{V}$	Vector fields appearing as counterterms	8
$\mathscr{V}$	Linear span of $\mathfrak{V}$	8
$\mathscr{V}^{\mathrm{nice}}$	Subspace of $\mathscr{V}$ consisting of 'minimal' counterterms	9
$\mathscr{V}_\star$	Subspace of $\mathscr{V}^{\mathrm{nice}}$ generated by $\tau_\star$	9
$\mathscr{V}_\star^\perp$	An arbitrary complement of $\mathscr{V}_\star$ in $\mathscr{V}^{\mathrm{nice}}$	9

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